ON CERTAIN COMPLETE CONFORMALLY FLAT SPACES
WITH CONSTANT SCALAR CURVATURE

By U-HANG KI AND GEHWAN OH

§ 0. Introduction.

Recently Yano, Houh and Chen ([5]) studied intrinsic problems of a conformally flat space in terms of sectional curvatures with respect to a unit vector field, in detail. They proved the following.

THEOREM 0.1. Let \((M, g)\) be an \(n\)-dimensional Riemannian manifold with a unit vector field \(u^h\). Then the necessary and sufficient conditions for \((M, g)\) having the properties:

1. The curvature operator \(K_{ji}^h v^h w^i\) associated with two vectors \(v^h\) and \(w^h\) orthogonal to \(u^h\) annihilates \(u^h\);
2. Sectional curvature with respect to a section containing \(u^h\) is a constant;
3. Sectional curvature with respect to a section orthogonal to \(u^h\) is a constant;

are that the Riemann-Christoffel curvature tensor of \((M, g)\) has the form

\[ K_{ji}^h = \lambda (\delta^h_{j} g_{ji} - \delta^h_{i} g_{jk}) + \mu \left( (\delta^h_{j} u_j - \delta^h_{i} u_k) u_i + (u_{ik} g_{ji} - u_{jk} g_{ii}) u^h \right) \]

for some functions \(\lambda\) and \(\mu\), \(g_{ji}\) being the Riemannian metric of \((M, g)\). In this case, \((M, g)\) is a conformally flat space for \(n > 3\).

In the present paper, we investigate a conformally flat space with two mutually orthogonal unit vector fields and with similar types of conditions of sectional curvatures as those stated in Theorem 0.1. Our main result is appeared in Theorem 2.4.

§ 1. Certain conformally flat spaces with two unit vector fields.

Let \((M, g)\) be a Riemannian manifold with metric tensor \(g\), whose components are \(g_{ji}\) with respect to local coordinate \(\{x^i\}\), where, here and in the sequel the indices \(h, j, i, \ldots\) run over the range \([1, 2, \ldots, n]\). We denote by \(\{\lambda_{ji}\}\) the Christoffel symbols formed with \(g_{ji}\) and by \(\nabla\) the operator of covariant differentiation with respect to \(\{\lambda_{ji}\}\). If we denote by \(K_{ji}^h\) the Riemann-Christoffel curvature tensor of \((M, g)\), then the Ricci tensor and the scalar curvature are given respectively by \(K_{ji} = K_{ji}^g\), \(K = g^{ji} K_{ji}\), where \(g^{ji}\) are contravariant components of \(g\).

We define a tensor field \(L_{ji}\) of type \((0, 2)\) by

\[ L_{ji} = -\frac{1}{(n-2)} K_{ji} + \frac{K}{2(n-1)(n-2)} g_{ji}. \]

The Weyl conformal curvature tensor \(C_{ki}^h\) is then given by

\[ C_{ki}^h = K_{ki}^h + \delta^h_{k} L_{ji} - \delta^h_{j} L_{ki} + L_{k}^h g_{ji} - L_{j}^h g_{ki}, \]

where, \(L_{ki}^h = L_{kg} g^{h}.\)

An \(n\)-dimensional Riemannian manifold \((M, g)\) is conformally flat ([3], [4]) if and only if
\[ C_{kji}^h = 0 \text{ for } n > 3, \]

\[ \nabla_k L_{ji} - \nabla_j L_{ki} = 0 \text{ for } n = 3. \]

It is well known that (1.4) can be derived from (1.3) for \( n > 3 \).

In this section we prove

**Proposition 1.1.** Let \((M, g)\) be an \( n \)-dimensional conformally flat space \(( n > 3 )\) with mutually orthogonal unit vector fields \( u^k \) and \( v^k \) satisfying the following two conditions:

1. The curvature operator \( K_{kji}^h X^i Y^j \) associated with two vector fields \( X^k \) and \( Y^k \) orthogonal to \( u^k \) and \( v^k \) respectively annihilates \( u^k \) and \( v^k \);

\[ K_{kji}^h X^i Y^j u^k = 0, \quad K_{kji}^h X^i Y^j v^k = 0. \]

2. Sectional curvature \( K(u) \) with respect to a section containing \( u^k \) orthogonal to \( v^k \), and vice versa are same value, and \( K(u) \) is a constant. Then we have

\[ L_{ji} = \alpha g_{ji} + \beta (u_i u_j + v_j v_i) \]

for some functions \( \alpha \) and \( \beta \).

**Proof.** We take \( n - 2 \) linearly independent vectors \( B_a^h, \quad (a, b, c, = 1, 2, 3, \ldots, n - 2) \), orthogonal to given unit vectors \( u^k \) and \( v^k \) and let \( B^i, u_i, v_i \) be determined in such a way that \( (B_a^h, u^k, v^k)^{-1} = (B^i, u_i, v_i) \). Then we have

\[ B_a^h B_i^b = \delta_a^i - u_i u^h - v_i v^h, \]

\[ B_a^h u_a = 0, \quad B_a^h v_a = 0. \]

The condition (1) is expressed as

\[ K_{kji}^h B_a^h u^i v^j = 0, \]

\[ K_{kji}^h B_a^h v^i u^j = 0, \]

\[ K_{kji}^h B_a^h B_i^b v^j = 0, \]

\[ K_{kji}^h B_a^h B_i^b u^j = 0. \]

Transvecting \( B_m^i \) to (1.8) and using (1.6), we find

\[ (\delta_m^h - u_m u^h - v_m v^h) K_{kji}^h u^i v^j = 0, \]

or, equivalently

\[ K_{mji}^h u^i v^j = v_m K_{kji}^h u^i v^j, \]

from which, transvecting with \( u^h \),

\[ K_{mji}^h u^i v^j u^h = - \mu v_m, \]

where \( \mu = -K_{kji}^h v^i u^j u^h. \)
On certain complete conformally flat spaces with constant scalar curvature

Similarly, transvecting (1.9) with $v^hB_m$, we have

\[(1.14) \quad K_{mij}v^i u^j = - \mu u^m.\]

Comparing with (1.12) and (1.14), we obtain

\[(1.15) \quad K_{mij}v^i u^j = - \mu v^m u_h.\]

From the second part of condition (II), we have $K_{kji}X^ku^k = \text{constant}$ for any unit vector $X^k$ orthogonal to $u^h$. This fact can be written as

\[(1.16) \quad K_{kji}u^kB_i = K(\sigma) g_{ij}.\]

If we transvect $B^m B^i$ to (1.16) and take account of (1.6) and (1.15), then the left hand side becomes

\[(1.17) \quad B^m B^i K_{kji} u^k u^i B^j B^h = K_{kml} u^k u^h - \mu v^mv_m.\]

And consequently (1.16) implies

\[(1.18) \quad K_{kji} u^k u^h - \mu v^i v_i = K(\sigma) (g_{ji} - u^i u^i - v^i v_i)\]

because of (1.6).

We can get from (1.2) and (1.3),

\[(1.19) \quad K_{kji} = - L_{khi} g_{ji} + L_{kij} g_{jh} - L_{jih} g_{kh} + L_{jhi} g_{ki}.\]

Transvecting (1.19) with $u^i v^j$ and using (1.15), we get

\[(1.20) \quad - \mu u^h v_k = - L(u, v) g^h + (L(u v^j) u^j + (L u v^j) v^j) v_k,\]

from which, transvecting with $g^{jk}$,

\[(1.21) \quad L(u, v) = 0\]

for $n > 2$, where $L(u, v)$ is defined by $L_{ji} u^i v^j$. Thus (1.20) implies that

\[(1.22) \quad L_{ji} u^i = L(u, u) u^i, \quad L_{ji} v^i = L(v, v) v^i,\]

\[(1.23) \quad L(u, u) + L(v, v) = - \mu.\]

If we transvect $u^h u^k$ to (1.19) and using (1.22), then we obtain

\[(1.24) \quad K_{kji} u^h u^k = - L_{ji} + L(u, u) (- g_{ji} + 2 u^j u^i).\]

From (1.18) and (1.24), we may have

\[(1.25) \quad L_{ji} = - \{K(\sigma) + L(u, u)\} g_{ji}\]

\[+ \{K(\sigma) + 2L(u, u)\} u^j u^i + \{K(\sigma) - \mu\} v^j v_i.\]

The first assumption of condition (II) means that
from which, transvecting $B^a_{\cdot m}B^b_i$ and taking account of (1.17),

$$K_{kmjkl}u^h v^h = K_{kmjkl}B^a_{\cdot m}B^b_i u^h v^h,$$

or. using (1.6) and (1.15),

$$K_{kmjkl}u^k h - \mu v^h v_m = K_{kmjkl}T^h u^h - \mu m h t.$$

Substituting (1.19) and (1.24) into the last equation and using (1.22), we have

$$-L_{ji} + L(u, u)(-g_{ji} + 2u_j u_i) - \mu u_j v_i$$

$$= -L_{ji} + L(v, v)(-g_{ji} + 2v_j v_i) - \mu u_j u_i,$$

from which, transvecting $g_{ji}$ and using (1.23),

$$L(u, u) = L(v, v), \quad 2L(u, u) = -\mu.$$

Thus (1.25) becomes

$$L_{ji} = \alpha g_{ji} + \beta (u_j u_i + v_j v_i).$$

where we have put

$$\alpha = \frac{1}{2} \mu - K(\sigma), \quad \beta = K(\sigma) - \mu.$$

This completes the proof of the proposition.

§ 2. A theorem on a complete conformally flat space with constant scalar curvature.

In this section, we consider a complete conformally flat space with constant scalar curvature satisfying all assumptions as those stated in Proposition 1.1.

Substituting (1.27) into (1.19), we find

$$(2.1) \quad K_{ijh} = 2\alpha (g_{ij} h - g_{h i j})$$

$$+ \beta \{ g_{ik}(u_j u_k + v_j v_k) - g_{ji}(u_k u_h + v_k v_h)$$

$$+ g_{jh}(u_i u_k + v_i v_k) - g_{ki}(u_j u_i + v_j v_i)\},$$

from which transvecting with $g_i^h$,

$$(2.2) \quad K_{ji} = 2 \{(1-n) \alpha - \beta\} g_{ji} + (2-n) \beta (u_j u_i + v_j v_i),$$

and consequently

$$(2.3) \quad K = -2(n-1)(n\alpha + 2\beta).$$
On certain complete conformally flat spaces with constant scalar curvature

Transvecting (2.2) with \( u^t \) and \( v^t \) respectively, we can easily see

\[
(2.4)\quad K_{jt}u^t = [2(1-n)\alpha - n\beta] u_j,
K_{jt}v^t = [2(1-n)\alpha - n\beta] v_j.
\]

If we transvect (2.2) with \( K_i^t \) and using (2.2) and (2.4), then

\[
(2.5)\quad K_{ji}K_{jt} = [4(1-n)\alpha - (n+2)\beta] K_{jk}
- 2[2(n-1)\alpha + n\beta] \{(n-1)\alpha + \beta\} g_{jk},
\]

which implies that

\[
(2.6)\quad K_{ji}K_{jt} = 2(n-1)(n\alpha + 2\beta) [4(1-n)\alpha + (n+2)\beta]
- 2n [2(n-1)\alpha + n\beta] \{(n-1)\alpha + \beta\}
\]

by virtue of (2.3).

Transvecting (2.5) with \( K^i^k \) and using (2.3) and (2.6), we find

\[
(2.7)\quad K_{ji}K_{ki}K^{jk}
= [4(1-n)\alpha - (n+2)\beta] [2(n-1)(n\alpha + 2\beta)]
\times [4(n-1)\alpha + (n+2)\beta]
- 2n [2(n-1)\alpha + n\beta] \{(n-1)\alpha + \beta\}
\times [4(n-1)(n\alpha + 2\beta) \{(n-1)\alpha + \beta\} [2(n-1)\alpha + n\beta].
\]

We see from (2.5) that the eigenvalue \( \lambda \) of \( K_i^\lambda \) satisfies the quadratic equation

\[
\lambda^2 + [4(n-1)\alpha + (n+2)\beta] \lambda
+ 2[2(n-1)\alpha + n\beta] \{(n-1)\alpha + \beta\} = 0,
\]

that is,

\[
(2.8)\quad \lambda_1 = -2(n-1)\alpha - n\beta, \quad \lambda_2 = -2(n-1)\alpha - 2\beta.
\]

**Lemma 2.1.** In a conformally flat space of dimension \( n \) with constant scalar curvature \( K \), we have (cf. See [6])

\[
(2.9)\quad \frac{1}{2} \mathcal{J}(K_{ji}K^{ji}) =
\frac{n}{n-2} K^i_i K_{ij} K^j_i - \frac{2n-1}{(n-1)(n-2)} K K_{ji} K^{ji}
+ \frac{1}{(n-1)(n-2)} K^3 - (\mathcal{P}^j_i K_{jk})(\mathcal{P}^j_k K^{jk}).
\]

Substituting (2.3), (2.6) and (2.7) into (2.9), we find the relationship
\[ \frac{1}{2} \Delta (K_{ij}K^{ij}) = -\frac{n}{n-2} \left[ 4(1-n)\alpha - (n+2)\beta \right] (2n-1) (n\alpha + 2\beta) \]
\[ \times \left\{ 4(n-1)\alpha + (n+2)\beta - 2n [2(n-1)\alpha + n\beta] \left( (n-1)\alpha + \beta \right) \right\} \]
\[ + 4(n-1) (n\alpha + 2\beta) \left( (n-1)\alpha + \beta \right) [2(n-1)\alpha + n\beta] \]
\[ - \frac{2n-1}{(n-1)(n-2)} \left\{ -2(n-1) (n\alpha + 2\beta) \right\} \]
\[ \times \left\{ 4(n-1)\alpha + (n+2)\beta - 2n [2(n-1)\alpha + n\beta] \left( (n-1)\alpha + \beta \right) \right\} \]
\[ + \frac{1}{(n-1)(n-2)} \left\{ -2(n-1) (n\alpha + 2\beta) \right\}^3 + (\nabla_j K_{ik}) (\nabla^i K^{jk}) \]

or, equivalently

\[ (2.10) \quad \frac{1}{2} \Delta (K_{ij}K^{ij}) = -2(n-2)^3 (2\alpha + \beta) \beta^2 - (\nabla_j K_{ik}) (\nabla^i K^{jk}). \]

**Lemma 2.2.** Let \((M, g)\) be a conformally flat space of dimension \(n \geq 3\) such that conditions (I) and (II) of Proposition 1.1 are satisfied, and the scalar curvature \(K\) is constant, then \(\alpha\) and \(\beta\) are both constants on \((M, g)\).

**Proof.** Differentiating (1.27) covariantly, we have

\[ (2.11) \quad \nabla_k L_{ji} = (-\frac{2}{n} g_{ji} + u_j u_i + v_j v_i) \beta_k \]
\[ + \beta \left\{ (\nabla_{k} u_j) u_i + (\nabla_{k} v_j) v_i + u_j (\nabla_{k} u_i) + v_j (\nabla_{k} v_i) \right\}, \]

where \(\beta_k = \Gamma_{kij}\).

On the other hand, we have from (1.1)

\[ (2.12) \quad \nabla_j L_{ij} = 0 \]

because of \(K = \text{constant}\).

Transvecting (2.11) with \(g^{ik}\) and using (2.12), we have

\[ (2.13) \quad \left\{ -\frac{2}{n} \beta_i + (u^i \beta_i) u_i + (v^i \beta_i) v_i \right\} \]
\[ + \beta \left\{ u_i (\nabla_i u^i) + v_i (\nabla_i v^i) + u^i (\nabla_i u_i) + v^i (\nabla_i v_i) \right\} = 0. \]

Taking skew-symmetric parts of (2.11) with respect to \(k\) and \(j\) and making use of (1.4), we obtain

\[ (2.14) \quad \left\{ (-\frac{2}{n} g_{ji} + u_j u_i + v_j v_i) \beta_k - (-\frac{2}{n} g_{ki} + u_k u_i + v_k v_i) \beta_j \right\} \]
\[ + \beta \left\{ (\nabla_{k} u_j - \nabla_{j} u_k) u_i + (\nabla_{k} v_j - \nabla_{j} v_k) v_i \right\} \]
\[ + u_j \nabla_{k} u_i - u_k \nabla_{j} u_i + v_j \nabla_{k} v_i - v_k \nabla_{j} v_i \right\} = 0 \]
On certain complete conformally flat spaces with constant scalar curvature

from which, transvecting with $w^i w^j$ and $v^i v^j$ respectively,

\begin{align}
(2.15) & \quad \frac{n-2}{n} \{ \beta_k - (w^i \beta_i) w^k \} + \beta \{ -u^i \nabla_i u_k - u^i u^j (\nabla_i v_j) v_k \} = 0, \\
(2.16) & \quad \frac{n-2}{n} \{ \beta_k - (v^i \beta_i) v_k \} + \beta \{ -v^i \nabla_i v_k - v^i v^j (\nabla_i u_j) u_k \} = 0.
\end{align}

Transvecting (2.15) with $v^k$ and (2.16) with $u^k$, we have respectively

\begin{align}
(2.17) & \quad \nabla^i \beta_i = 0, \\ & \quad u^i \beta_i = 0.
\end{align}

Substituting (2.15) and (2.16) into (2.13) and taking account of (2.17), we find

\begin{align}
(2.18) & \quad \frac{2(n-3)}{n} \beta_k = A u_k + B v_k
\end{align}

for suitable functions $A$ and $B$, and consequently $\beta$ is constant by virtue of (2.17).

Since $K$ and $\beta$ are both constants, $\alpha$ is also because of (2.3). Thus Lemma 2.2 is proved.

**Lemma 2.3.** Under the same assumptions as those stated in Lemma 2.2, we have $\beta = 0$ or $2\alpha + \beta = 0$ on $(M, g)$.

*Proof.* We have from (2.5) and Lemma 2.2

\begin{align}
(2.19) & \quad K_{ij} K_{ij} = a K_{ij} + b g_{ij}
\end{align}

for some constants $a$ and $b$, where

\begin{align}
(2.20) & \quad a = 4(1-n)\alpha + (n+2)\beta, \\
& \quad b = -2 [2(n-1)\alpha + n\beta] \{(n-1)\alpha + \beta\}.
\end{align}

On the other hand, we see that

\begin{align}
(2.21) & \quad \nabla^i K_{ij} - \nabla^j K_{ij} = 0
\end{align}

because of (1.1), (1.4) and $K =$ constant.

Differentiating (2.19) covariantly, we get

\begin{align}
(2.22) & \quad (\nabla^i K_{ij}) K_{ij} + K_{ij} (\nabla_i K_{ij}) = a \nabla^i K_{ij},
\end{align}

from which, taking skew-symmetric parts with respect to $k$ and $j$ and using (2.21),

\begin{align}
K_{ij} (\nabla^i K_{ij}) - K_{ij} (\nabla^j K_{ij}) = 0,
\end{align}

or, changing $k$ with $i$,

\begin{align}
(2.23) & \quad K_{ij} (\nabla^i K_{ij}) - K_{ij} (\nabla^j K_{ij}) = 0.
\end{align}

Adding (2.22) to (2.23), we find

\begin{align}
2 K_{ij} (\nabla^i K_{ij}) = a \nabla^i K_{ij},
\end{align}
from which, transvecting $K_i^i$ and using (2.19),

$$aK_t^rK_t^r + 2bK_t^rK_t^r = 0.$$ 

Thus the last two equations mean that

$$(a^2 - 4b)\nabla^2 K_{ij} = 0.$$ 

Since $a$ and $b$ are both constants, we have $a^2 - 4b = 0$ or $\nabla^2 K_{ij} = 0$ on $(M, g)$. If $a^2 - 4b = 0$, then we easily verify that $\beta = 0$ because of (2.8) and (2.20). If $\nabla^2 K_{ij} = 0$, then $\Delta(\nabla^2 K_{ij}) = 0$. In this case, we see $\beta = 0$ or $2\alpha + \beta = 0$ by virtue of (2.10). This complete the proof.

Summing up the arguments developed above, if $\beta = 0$, $(M, g)$ is a real space form because of (2.1). If $2\alpha + \beta = 0$, then the eigenvalues of the Ricci tensor $K_{ij}$ are $2\alpha$ or $-2\alpha (n-3)$ by virtue of (2.8). In usual way (cf. See Ryan [1], Sekigawa and Tagagi [2]), owing to completeness, $(M^g, g)$ is one of $E^n$ or $E^2 \times S^{n-2}(c)$.

We are now in a position to have a theorem.

**Theorem 2.4.** Let $(M^g, g)$ $(n > 3)$ be a complete conformally flat space with mutually orthogonal unit vector fields $u^h$ and $v^h$ and with constant scalar curvature satisfying the two conditions:

1. The curvature operator $K_{ij}^k X^i Y^j$ associated with two vector fields $X^h$ and $Y^h$ orthogonal to $u^h$ and $v^h$ respectively annihilates $u^h$ and $v^h$.

2. Sectional curvature $K(a)$ with respect to a section containing $u^h$ orthogonal to $v^h$, and vice versa are same value, and $K(a)$ is a constant. Then $(M^g, g)$ is one of $S^n(c)$, $E^n$ or $E^2 \times S^{n-2}(c)$, the real space forms of curvature $c$ being denoted by $S^n(c)$ or $E^n$ depending on whether $c$ is positive or zero.

**Bibliography**


Kyungpook University