1. Introduction. B. Malgrange introduced in [1] $P(D)$-related Frechet spaces to prove $P(D)\mathcal{D}^\infty F(\Omega) = \mathcal{D}^\infty F(\Omega)$ where $P(D)$ is a differential polynomial and $\mathcal{D}^\infty F(\Omega)$ is the space of distributions in $\Omega$ of finite order. The functional analytic version of $P(D)$-related Frechet spaces and its generalizations are developed in F. Treves [2]. The central result in [2] is the corollary 2 of theorem 17.2; it reads in our notations (cf.2), if the map $T$ from $E$ into $F$ is such that the image of $T$ is weakly closed in $E'$ and $T$ is injective, then $T$ from $G$ into $M$ is an epimorphism and has a homogeneous approximation property iff $L$ and $M$ are $T$-related. In this paper we weaken the condition for $T$ and assuming the condition for $T$ to be presurjective we get the similar results. The methods used here clarifies the connections between presurjectivity of $T$ and $T$-related Frechet spaces.

I thank the referee for revising the article.

2. Notations and Definitions. Let $E$ and $F$ be locally convex topological linear spaces. We further assume that $F$ is barrelled. Let $\Sigma$ be a locally convex Hausdorff topological space. Let $L$ and $M$ be Frechet spaces. We assume that

i) $E(F)$ is continuously imbedded in $L$ ($M$ resp.) and dense in $L$ ($M$ resp.), and

ii) $L$ and $M$ are continuously imbedded in $\Sigma$.

Let $T$ be a continuous transformation from $\Sigma$ into $\Sigma$ such that $T$ restricted on $E$ (which we shall denote by $T$ again) is a continuous transformation from $E$ into $F$. We shall denote by $E'$ the continuous dual of $E$ and by $T'$ the continuous transpose of $T$ as usual. For any $f \in E'$, $|f|(x) = |f(x)|$ for every $x \in E$. For any seminorm $q$ on $F$, $Tq$ is a seminorm on $E$ such that $Tq(x) = q(T(x))$. We shall use the following definitions.

**Definition 1.** A continuous map $T$ from $E$ into $F$ is presurjective iff for any continuous seminorm $p$ on $E$ there exists a continuous seminorm $q$ on $F$ such that for any $g \in F'$, $Tq \leq p$ implies $|g| \leq q$.

**Definition 2.** $L$ and $M$ are $T$-related iff for any $g \in F'$, $Tg \in L'$ implies $g \in M'$.

**Definition 3.** A continuous map $T$ from $E$ to $F$ is an epimorphism iff it is surjective and open.

**Definition 4.** The map $T$ from $L$ into $\Sigma$ has a homogeneous approximation property iff for any $x$ in $L$ such that $Tx = 0$ there exists a sequence $\{x_i\}$ of the elements in $E$ such that $Tx_i = 0$ and $\{x_i\}$ converges to $x$ in $L$.

3. Theorems. Let $G$ be the linear subspace of $L$ consisting of those $x$ such that $Tx \in M$. We can identify $G$ with the subset of $L \times M$ consisting of the pairs $(x, Tx)$, $x \in G$. We provide $G$ with the topology induced by the product topology. Then $G$ is a Frechet space; it is enough to prove that the set $\{(x, Tx) | x \in G\}$ is closed in $L \times M$. If $\{x_i\}$ is a sequence converging to $x$ in $L$ and such that $Tx_i$ converges to $y$ in $M$, since $L$ is con
continuously imbedded in $\Sigma$, $Tx_i$ must converge to $Tx$ in $\Sigma$, hence $y=Tx$ since $\Sigma$ is Hausdorff. We note that $G$ is continuously imbedded in $L$ and $T$ from $G$ to $M$ is continuous.

**Theorem 1.** If the map $T$ from $E$ to $F$ is presurjective and $G$ and $M$ are $T$-related, then $T(L) \supseteq M$.

**Proof.** We shall show that $T(G)=M$ from which $T(L) \supseteq M$ follows. Since $T(G) \subset M$ by definition and since $G$ and $M$ are Frechet, to show that $T(G)=M$, it is enough to show that the map $T$ from $G$ to $M$ is presurjective (cf. [2]). Let $p$ be a continuous seminorm on $G$. We may identify $p$ with a continuous seminorm on $E$. Since $T$ from $E$ to $F$ is presurjective, there exists $q$, a continuous seminorm on $F$ such that for any $f \in F'$ $|Tf| \geq p$ implies $|f| \leq q$. Let $q'=\sup\{|f| f \in F' \text{ and } |Tf| \leq p\}$. Then $q'$ is a continuous seminorm on $F$ since $q' \leq q$. We note that $Tq' \leq p$, that is, for any $x \in E$ $q'(Tx) \leq p(x)$. Let $g \in F'$ such that $|g| \leq q'$. Then $|Tg| \leq p$. Therefore $Tg \in G$. Since $G$ and $M$ are $T$-related, this shows that $g \in M$. Now since $M$ is a Frechet space, and since for any $g \in F'$ such that $|g| \leq q'$ $g \in M'$, this implies $q'$ is a continuous seminorm on $M$ (cf. p. 48[2]). Above arguments shows that for any continuous seminorm $p$ on $G$ there exists a continuous seminorm $q'$ on $M$ such that for any $g \in M'$ $|Tg| \leq p$ implies $|g| \leq q'$. Therefore $T$ from $G$ to $M$ is presurjective and hence is an epimorphism.

**Corollary.** If the map $T$ from $E$ to $F$ is presurjective, $T(L) \subset M$, $T$ from $L$ into $M$ is continuous, and $L$ and $M$ are $T$-related, then $T(L)=M$.

**Proof.** We note that the space $G$ introduced before is topologically equivalent to $L$. Hence our corollary follows immediately from the previous theorem.

**Theorem 2.** Assume that $T(E)$ is dense in $F$, $T(L) \supseteq M$ and the map $T$ from $L$ to $M$ has a homogeneous approximation property. Then $L$ and $M$ are $T$-related.

**Proof.** Let $g \in F'$ be such that $Tg \in L'$. On $M$ define a linear functional $h$ such that for any $y \in M$ $h(y)=Tg(x)$ where $x \in L$ is such that $Tx=y$. Then $h(y)$ does not depend on $x$. For if $x' \in L$ is also such that $Tx'=y$, we have $T(x-x')=0$. Since $T$ has an homogeneous approximation property, there exists a sequence $\{x_i\}$ in $E$ converging to $x-x'$ in $L$ and such that $Tx_i=0$ for all $i$. We then have $Tg(x)-Tg(x')=\lim_Tg(x_i)$ for all $i$. Let us go back to the space $G$ introduced before. Since $G$ and $M$ are Frechet spaces, $T$ is an open mapping from $G$ onto $M$. Therefore if $y_i \to 0$ in $M$, there exists, for every $i$, $x_i \in G$ such that $x_i \to 0$ in $G$, a fortiori in $L$, and such that $Tx_i=y_i$. But then $h(y_i)=Tg(x_i) \to 0$. This proves the continuity of $h$. Hence $h \in M'$. Since $h(y)=Tg(x)$ for any $y \in M$, take in particular $y \in T(E) \subset F$. Then there is $x \in E$ such that $Tx=y$. This yields for any $y \in T(E)$ $h(y)=Tg(x)=g(y)$. Since $T(E)$ is dense in $M$, we conclude that $h=g \in M'$.

**Corollary.** Assume that the map $T$ from $E$ into $F$ is presurjective, $T(L) \supseteq M$, and $T$ has a homogeneous approximation property, then $L$ and $M$ are $T$-related.

**Proof.** Since $T$ is presurjective, $T(E)$ is dense in $F$. Hence the previous theorem completes the proof.
References


Seoul National University