A NOTE ON \(K_0(I)\)

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Let \(I\) be a two sided ideal in an associative ring \(R\) with unit. Then there is an exact sequence of abelian groups:

\[
K_1(R) \to K_1(R/I) \to K_0(I) \to K_0(R) \to K_0(R/I)
\]

of length five.

For the definitions of the group \(K_0(I)\), the functors \(K_0, K_1\) and for the proof see Milnor [2].

In the case that \(R\) is the ring of integers in a finite extension field of rational numbers the following theorem is to be proved, in this note, and for certain ideals \(I\) in the rings \(\mathbb{Z}, \mathbb{Z}(\sqrt{-5})\), \(K_0(I)\) are to be computed.

As a matter of notation, \(U(A)\) denotes the group of units for an associative ring \(A\). For a group homomorphism \(h\), \(\ker(f) \ (\text{coker}(f))\) denotes the kernel (the cokernel) of \(f\).

**THEOREM.** If \(I\) is an ideal of the ring \(R\) of integers in a finite extension field of rational numbers. Then we are given a short exact sequence of abelian groups.

\[
0 \to \text{coker}(U(R) \to U(R/I)) \to K_0(I) \to \text{cl}(R) \to 0.
\]

In particular, we have

\[
K_0(I) \cong \text{coker}(U(R) \to U(R/I)) \oplus \text{cl}(R)
\]

if the orders of the abelian groups, \(\text{coker}(U(R) \to U(R/I)), \text{cl}(R)\) are relatively prime.

**Proof.** Since \(R\) is the ring of integers in a finite extension field of rational numbers the ideal class group \(\text{cl}(R)\) of \(R\) is finite and \(K_1(R) \cong U(R)\), moreover we have \(K_0(R) \cong \mathbb{Z} \oplus \text{cl}(R)\) [2].

On the other hand, if \(I\) is a non-zero ideal in the ring \(R\), then \(R/I\) is finite [2], and so \(K_0(R/I)\) is finitely generated free abelian group, since \(R/I\) is an Artin ring [4]. Furthermore, we have \(K_1(R/I) \cong U(R/I)\), since \(R/I\) is semilocal Milnor [2].

Now note the following exact sequence of abelian groups, which is induced from the exact sequence (1):

\[
U(R) \to U(R/I) \to K_0(I) \to K_0(R) \to K_0(R/I)
\]

where \(U(R) \to U(R/I)\) is the canonical group homomorphism.

\(K_0(R/I)\) is torsion-free, \(K_0(R) \cong \mathbb{Z} \oplus \text{cl}(R)\) and the homomorphism \(K_0(R) \to K_0(R/I)\) is not the zero homomorphism. Therefore \(\ker(K_0(R) \to K_0(R/I))\) is isomorphic to the ideal class group \(\text{cl}(R)\).

Received by the editors Aug. 26, 1975.
Thus, the short exact sequence (2) follows now immediately from the exact sequence (4). This completes the proof.

The following simple lemma is needed to compute \( K_0(I) \).

**Lemma.** Let \( \mathbb{Z} \) be the ring of integers. Let \( p \) be a prime number \( \geq 3 \). Then the group \( U(\mathbb{Z}/p^n\mathbb{Z}) \) of units is a cyclic group of order \( p^{n-1}(p-1) \) for \( n \geq 1 \). For \( p=2 \), we have that \( U(\mathbb{Z}/2^n\mathbb{Z}) \) is an abelian group of type \((2^{n-2}, 2)\) for \( n \geq 3 \) Speiser [3].

Let \( I \) be the ideal \( 2^n\mathbb{Z} \) in the ring \( \mathbb{Z} \), where \( n \geq 3 \). Then \( U(\mathbb{Z}/I) \) is an abelian group of type \((2^{n-2}, 2)\) by the Lemma. On the other hand \( U(\mathbb{Z}) = \{1, -1\} \), \( \text{cl}(\mathbb{Z}) = 0 \) Hence \( K_0(2^n\mathbb{Z}) \) is a cyclic group of order \( 2^{n-2} \) by the exact sequence (2) The following two results are obvious, \( K_0(2\mathbb{Z}) = 0 \), \( K_0(2^2\mathbb{Z}) = 0 \).

Let \( R \) be the ring \( \mathbb{Z}(\sqrt{-5}) \). Let \( I \) be the principal ideal \((2-\sqrt{-5})\) in the ring \( \mathbb{Z}(\sqrt{-5}) \). Then we have \( U(R) = \{1, -1\} \), \( \text{cl}(R) \) is a group of order 2, and \( R/I \) is a cyclic group of order 3, and so \( \text{coker}(U(R) \to U(R/I)) \) is a cyclic group of order 3. Hence

\[
K_0((2-\sqrt{-5})) \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}
\]

where \( \mathbb{Z}/3\mathbb{Z} \), \( \mathbb{Z}/2\mathbb{Z} \) denote the additive cyclic groups.

**References**


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