A BILLIARD TABLE PROBLEM IN $E^n$

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To understand a general case in $E^n$ we begin with a familiar billiard table problem in $E^2$.

1. In $E^2$. We consider first a billiard table and a shot satisfying the following three conditions: (i) The table dimension is $a \times b$ where $a$ and $b$ are positive integers and relatively prime. (ii) One billiard ball is shot from the lower left-hand corner at a $45^\circ$ angle to its sides. (iii) The ball travels indefinitely, unless it hits a corner in which case it stops.

**Theorem 1.** The ball stops after traveling a total distance of $\sqrt{2}ab$ while striking $a \cdot b - 2$ cushions, excluding the two corners of its departure and termination. If we name the four corners 0, 1, 2, and 3 as in Figure 1 ($a=2$, $b=5$), the ball will terminate at corner 1 if $a$ is even, 2 if $b$ is even, and 3 if both $a$ and $b$ are odd.

**Proof.** Consider a lattice in the first quadrant tessilated by $a \times b$ rectangles as in Figure 2.
By reflection, there is a 1-1 correspondence between the line $x=y$ and the ball’s actual path as in Figure 1, and between the intersections of the line $x=y$ with the lattice and the cushions struck by the ball. The line $x=y$ first intersects a lattice point $L$ at $(1,1)$, corresponding to the terminal corner, where $l$ is the l.c. m. of $a$ and $b$ and hence $l=ab$, and en route intersects $l/a-1(-b-1)$ vertical and $l/b-1(-a-1)$ horizontal lattice lines. Accordingly, the ball strikes $a+b-2$ cushions before it strikes the corner.

To identify the corner which corresponds to the point $L$, we represent each reflection of the table by permutations $P_1=(01)(23), P_2=(02)(13)$ and their combinations $P_1P_2=P_2P_1=(03)(12)$, $I$ the identity. In particular, the last table position whose corners contain $L$ is given by $P_1^{a-1}P_2^{b-1}$ because the table is being reflected along the $b$-side $b-1$ times and the $a$-side $a-1$ times, and the terminal corner can be found as a number which replaces the corner 3 by $P_1^{a-1}P_2^{b-1}$. Since

$$P_1^{a-1}P_2^{b-1} := \begin{cases} \text{if } a \text{ is even}, P_2 = (02)(13) & \text{if } a \text{ is even}, \\ \text{if both } a \text{ and } b \text{ are odd}, & P_1 = (01)(23) \end{cases}$$

the terminal corner is 1 if $a$ is even, 2 if $b$ is even, or 3 if both are odd, as stated in the theorem. Note that $P_1P_2$ does not occur in the final position.

To extend the above theorem beyond the conditions of standardization described at the beginning, we may consider the case $(a,b)=t(a',b')$ where $t>1$ and $(a',b')=1$. Then the number of cushions and the terminal corner on the table $a\times b$ can be found from a similar table of reduced size $a'\times b'$, but the total distance traveled will be $\sqrt{2}ab/t (=t(\sqrt{2}a'b'))$ on the table $a\times b$. Secondly, we change the direction of the ball. If the ball is shot toward a point $(1,r)$ for rational $r$, instead of $(1,1)$ or a 45° from the sides of the table $a\times b$, then consider a table of size $a'\times b'$ which is similar to a table of size $a\times (b/r)$ and $(ka,kb/r)=(a',b')=1$ for some rational $k$. Then the number of cushions is $a'+b'-2=k(a+b/r)-2$ and the length of the ball’s path is $(\sqrt{2}a'b'/k)(\sqrt{1+r^2}/\sqrt{2})=kab\sqrt{1-r^{-2}}/r$ in the original table $a\times b$.

2. In $E^3$. Here we will try to generalize theorem 1 in a higher dimension. In $E^3$, in particular, a ball’s path on a billiard table can be interpreted as a path of a light beam reflected on mirrored surfaces inlaid inside a rectangular
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box. Call a generalized rectangular box a hyper box in $E^*$ and name $2^n$ corners by numbers $0, 1, 2, \ldots, 2^n - 1$ in the following fashion: Take a corner as the origin and name it $0$ (zero). Then take $n$ edges from $0$ in an arbitrary order as the $x_1, x_2, \ldots, x_n$ axes. Suppose the size of our hyper box is $a_1 \times a_2 \times \cdots \times a_n$, where $a_i$ is a positive integer length along the $x_i$ axis and $(a_1, a_2, \ldots, a_n) = 1$. Then the coordinate of a corner is one of the expressions

$$P_i = (01)(23)\ldots(2^n-2, 2^n-1),$$

these $n$ permutations generate a commutative group of order $2^n$ and the final position of the reflected box will be $\Pi P_i^{l_i}$. Hence the terminal corner can be identified by the number replaced by $2^n - 1$ in the above final position.

Let $\Pi P_i^{l_i} = P_i^{l_i}$, where $l_i = 1$ or $0$ according to whether $\frac{l_i}{a_i} - 1$ is odd or even. And $\frac{l_i}{a_i} - 1$ is odd if $l_i/a_i$ is even. If we denote the exponent of 2 in the prime power representation of an integer $j$ by $|j|_2$ (e.g. $|40|_2 = |2^5|_2 = 3$), $l_i/a_i$ is even iff $|a_i|_2 < |l|_2$.

Hence

$$\epsilon_i = \begin{cases} 1 & \text{if } |a_i|_2 < |l|_2, \\ 0 & \text{if } |a_i|_2 = |l|_2. \end{cases}$$

Since the permutation $P_i^{l_i}$ causes replacement of $2^n - 1$ by $2^n - 1 - \epsilon_i 2^{n-1}$ and $P_i^{l_i}$ causes replacement of the latter by $2^n - 1 - \epsilon_i 2^{n-1} - 2^{n-2}$ again, and so on, the terminal corner will be

$$2^n - 1 - \sum_{i=1}^{n} \epsilon_i 2^{i-1} = \sum 2^{i-1} - \sum \epsilon_i 2^{i-1} = \sum_{i=1}^{n} (1 - \epsilon_i) 2^{i-1}.$$ 

Therefore, the corner we are looking for can be expressed

$$C = \sum_{i=1}^{n} \delta_i 2^{i-1}, \quad \text{where } \delta_i = \begin{cases} 1 & \text{if } |a_i|_2 = |l|_2, \\ 0 & \text{if otherwise.} \end{cases}$$
For instance, if all $a_i$'s are odd, $\prod_i P_i = 1$, the identity and $C = \sum_i 2^{i-1} = 2^n - 1$. Among the $2^n$ elements of the group the permutation $P_1P_2\cdots P_n$ which would replace $2^n - 1$ by 0 never occurs in the final position because $\frac{l}{a_i} - 1$ cannot be all odd i.e. $l/a_i$ cannot be all even to satisfy $(a_i, a_2, \ldots, a_n) = 1$.

When the ball is hit toward the point $(1, 1, \ldots, 1)$ from 0 (the distance of the two points is $\sqrt{n}$), the ball will pass the points $(1, 1, \ldots, 1)$, $(2, 2, \ldots, 2)$, ..., until it hits a side $(n-1$ dimensional box) and reflects there.

Summarizing the above argument we state the following theorem.

We name this corner by the number

$$e_1 + e_2 + e_3 + e_4 + \cdots + e_n 2^{n-1},$$

one of $0, 1, 2, \ldots, 2^n - 1$. Figures 3 and 4 show boxes with properly named corners. In this setting the equation of the initial direction of a ball's path would be $x_1 = x_2 = \cdots = x_n$ and the number of cushions would be $\sum_{i=1}^{n} (l/a_i - 1)$, where $l$ is the l.c.m. of the $a_i$'s. Since at every cushion our hyper box is reflected and its reflected position can be expressed by a combination of the following $n$ permutations of order 2:

**THEOREM 2.** In $E^n$, the ball stops after traveling a total distance of $\sqrt{nl}$ while striking the sides of a hyper box of size $a_1 \times a_2 \times \cdots \times a_n \sum_{i=1}^{n} (l/a_i - 1)$.
times, excluding the two corners of its departure and termination. If we name the corner \((\varepsilon_1a_1, \varepsilon_2a_2, \ldots, \varepsilon_na_n)\) with \(\varepsilon_i=0\) or \(1\) by \(\sum\varepsilon_i2^{i-1}\), then the ball will terminate at the corner \(\sum_{i=1}^n \delta_i2^{i-1}\), where \(\delta_i\) is defined by 
\[
\delta_i = \begin{cases} 
1 & \text{if } |a_i|_2 = |I|_2, \\
0 & \text{otherwise.}
\end{cases}
\]

**Remarks.** 1. In case \(a_1, a_2, \ldots, a_n\) are pairwise relatively prime, the distance traveled by the ball is \(\sqrt{n}a_1a_2\cdots a_n\) and this implies that the ball has to pass through every unit hyper box \(a_1 \times a_2 \times \cdots \times a_n\) once and only once. 2. In figure 1 if we que a ball from the corner 2, it will terminate at 3. An interesting fact is that if we shoot a ball from any point on the table but not on the two known paths from a corner to another, then it will travel a loop of length exactly equal to two times that of the ball’s path from corner to corner. (Shoot the ball parallel to the usual direction.)

**References**


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