A NOTE ON THE ESSENTIAL NILPOTENCY

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An ideal $L$ of a ring $R$ is called essentially nilpotent if it contains a nilpotent ideal $N$ of $R$ which is essential in $L$, i.e., $N$ has non-zero intersection with each non-zero ideal of $R$ which is contained in $L$ [2]. A nil right (left) ideal of a ring is called essentially right (left) nilpotent if it contains a nilpotent right (left) ideal which is essential in it. If $N$ is an ideal of a ring $R$, $N$ is called left $T$-nilpotent ("$T$" for transfinite) if, given any sequence $\{a_i\}$ of elements in $N$, there exists an $n$ such that $a_0 a_1 \cdots a_{n-1} a_n = 0$. (Right $T$-nilpotency requires instead that $a_0 a_1 \cdots a_{n-1} a_n = 0$.) [1].

Shock proved that a nil right ideal is essentially nilpotent if and only if it contains an essential right ideal which is left $T$-nilpotent [3]. In this paper, we show that an ideal $L$ is essentially nilpotent if and only if $L$ contains a left $T$-nilpotent ideal which is essential in $L$.

REMARK 1. H. Bass' exemple (5), p. 476 of [1] shows the existence of a left $T$-nilpotent ideal but not right $T$-nilpotent. Therefore if an ideal $N$ is left $T$-nilpotent, then $N$ is not nilpotent.

REMARK 2. Using the Sasiada's example (Let $R$ denote the ring generated over the integers by $x_1, x_2, \cdots$ with the relation $x_i x_j = 0$ for $i \geq j$.), J. Fisher showed that essential nilpotency does not imply left $T$-nilpotency [2].

LEMMA. If an ideal $L$ of $R$ is left $T$-nilpotent, then $L$ is essentially nilpotent.

Proof. In [2].

THEOREM. An ideal $L$ of a ring is essentially nilpotent if and only if $L$ contains a left $T$-nilpotent ideal which is essential in $L$.

Proof. Let $J$ be a nilpotent ideal of $R$ which is essential in essentially nilpotent ideal $L$. Hence $J$ is a left $T$-nilpotent ideal which is essential in $L$.

Conversely, let $J$ be a left $T$-nilpotent ideal contained in $L$ which is essential in $L$. By the previous lemma, the left $T$-nilpotent ideal $J$ is essentially nilpotent. Hence $J$ contains a nilpotent ideal $N$ which is essential in $J$. $N$ is
also essential in $L$. Hence $L$ is essentially nilpotent.

References


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