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ON COMPACTNESS OF BITOPOLOGICAL SPACES

By Charles Byrne

Several authors have offered definitions of compactness for bitopological spaces. The paper of Cooke and Reilly presents each of these definitions, clears up some confusion that existed concerning the connections between certain of these definitions, and discusses the desirable and not so desirable aspects of each. No one definition has yet managed to imply all the properties we would like to associate with any definition of compactness. This paper offers yet another notion of compactness, d-compactness, and considers its position relative to the other definitions. Swart asks for a type of compactness which can be pairwise Hausdorff without the two topologies being identical, and which is also productive. As we shall see, d-compactness does not enjoy this property. However, it enjoys a weaker version, in that it is productive, and can be weakly pairwise Hausdorff without the topologies being the same, and there is reason to believe that pairwise Hausdorff is too strong a property to seek. In addition, d-compactness implies pseudo-compact and does not force the two topologies to be compact separately, thus combining two desirable properties that have not previously been enjoyed by the same type of compactness. All terms used above will be defined shortly, along with appropriate references.

DEFINITION 1. A cover of a bitopological space (X,U,V) is U,V *(pairwise) open* if its members come from U or from V (and if at least one non-empty member of each topology is used).

DEFINITION 2. (Fletcher, Hoyle, and Patty [4]) The bitopological space (X,U,V) is *pairwise compact* if every pairwise open cover has a finite subcover.

REMARK. Cooke and Reilly ([3]) have shown that this definition coincides with previous definitions given by Kim, and by Kim and Naimpally(see also [5]).

DEFINITION 3. (Swart [9]) The bitopological space (X,U,V) is *semi-compact* if every U,V open cover has a finite subcover.

REMARK. Semi-compactness is equivalent to compactness of X in the topology generated by U and V.

DEFINITION 4. (Birsan [1], [2]) The bitopological space (X, U, V) is *B-compact* if every U open cover has a finite subcover, that is also V open, and the same is true with the roles of U and V reversed.

DEFINITION 5. A mapping f from topological space (X, T) to the real line is T *upper semi-continuous* (*usc*) if the inverse image of sets of the form $(-\infty, a)$ is in T for every real a. We say f is T *lower semi-continuous* (*lsc*) if inverse images of sets $(a, +\infty)$ are in T. We say that (X, U, V) is *pseudo-compact* if every U usc, V lsc f from X to the real line is bounded.

DEFINITION 6. (Saegrove [7]) The bitopological space (X, U, V) is *bi-compact* if it is pseudo-compact and pairwise homeomorphic to the intersection of a $\prod R$ closed set and a $\prod L$ closed set in the product of copies of the real line, with R the topology of right-infinite open rays, L, the topology of left-infinite open rays.

DEFINITION 7. The bitopological space (X, U, V) is *pairwise Hausdorff* if, given any two points, x, y in X, there are disjoint sets A and B, with A in U, and B in V, A containing x, B containing y. We say the space is *weakly pairwise Hausdorff* if the above holds, except that we may not be able to specify which point goes in which kind of open set.

What is desired is a definition of compactness for a bitopological space (X, U, V) that a) is productive, b) can be enjoyed by a pairwise Hausdorff space without forcing the two topologies to coincide, c) does not imply that the spaces (X, U) and (X, V) be compact, and d) implies pseudo-compactness. No known example enjoys both a) and b). It is easily shown that b) implies c). Pairwise compactness has b) and c), but lacks the other two. The other three types of compactness, semi-compactness, B-compactness, and bi-compactness, all have a) and d), but fail to have b) and c). We shall show that our notion of d-compactness has all of the above properties, except b).

DEFINITION 8. Let U and V be topologies for X. We say (X, U, V) is *d-compact* if X contains a U compact, U closed, V dense subset, and a V closed, V compact, U dense subset.

REMARK. This is equivalent to compactness, when U and V are identical.

A space will be d-compact whenever the two topologies are compact hence whenever the bitopological space is semi-compact, B-compact or bi-compact.

An example below will show that a space may be d-compact without U and V

being compact, so d-compactness does not imply the above-mentioned three types of compactness. In addition, this example will not be pairwise compact. Another example will show that a pairwise compact space may not be d-compact.

PROPOSITION 1. *The product of d-compact spaces is again d-compact.*

PROOF. Let (X_i, U_i, V_i) be d-compact, K_i a U_i compact, V_i dense subset of X_i, and M_i a V_i compact, U_i dense subset of X. Then the product of the sets K_i is compact in the product topology formed by the U_i, and dense in the product topology formed by the V_i. Similarly, the product of the sets M_i serves as the other desired set.

The following example will help establish a number of our assertions concerning d-compact spaces.

EXAMPLE. Let X be the Stone-Cech compactification of N, the natural numbers with the discrete topology. Let U be the usual topology on X, with respect to which the subset N is open and dense. Let V be the collection of subsets A of X, with the property that $A \cap N$ is a member of the co-finite topology on N, the topology generated by subsets whose complement is finite. Then $K = X$ is U closed, U compact and V dense, while N is V closed, V compact and U (X, V) dense. So (X, U, V) is d-compact. This space is not separately compact, since is not a compact space. We therefore have

PROPOSITION 2. *A bitopological space may be d-compact without both topologies being compact, hence without being semi-compact, B-compact, or bi-compact.*

PROPOSITION 3. *A bitopological space may be d-compact without being pairwise compact.*

PROOF. Consider the pairwise open cover of X consisting of all the singleton sets. Those taken from N are U open, while those taken from $X-N$ are V open (since such a singleton has empty intersection with N, and the empty set is in the co-finite topology.)

The above example is not a pairwise Hausdorff space, since no element in $X-N$ can be contained in a U open set that has finite intersection with N (see the paper of Rudin for details concerning X). It is, however, weakly pairwise Hausdorff.

REMARK. In a recent paper of Smithson ([8]) it is shown that bitopological spaces that arise by taking a family of set-valued maps and generating the smallest topologies on the domain with respect to which the family is upper semi-continuous and lower semi-continuous (his definitions are not the ones above) will be weakly pairwise Hausdorff if and only if the family separated points, but that to get pairwise Hausdorff, a stronger condition than that is required. One might be persuaded to replace b) above with the weaker condition obtained by inserting weakly pairwise Hausdorff. With b) so modified, d-compactness enjoys all four of the desired properties.

We showed that d-compactness does not imply pairwise compactness. What about the converse ?

Example 4 of [4] provides a pairwise compact, pairwise Hausdorff space in which the two topologies are distinct. Such a space cannot be d-compact.

We conclude by showing that all d-compact spaces have d).

PROPOSITION 4. *Every d-compact space is pseudo-compact.*

PROOF. Let (X, U, V) be d-compact, K a U compact, V dense subset, M, a V compact, U dense subset. If f is U usc and V lsc, and not bounded above, then $f^{-1}(-\infty, n)$ is a U open cover of X that has no finite subcover. But, finitely many of these sets do cover K, so that for some positive m, $f^{-1}(m, +\infty)$ is disjoint from K. But then K is not V dense. A similar argument shows that f is bounded below.

The Catholic University of America
Washington, D.C. 20064

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