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ON DECOMPOSITION OF RECURRENT CURVATURE TENSOR FIELDS IN GENERALISED FINSLER SPACES

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Some aspects of generalised Finsler spaces were studied by A.C. Shamihoke [1] [1]. Author and Sinha [2] have defined recurrent generalised Finsler spaces and dealt with the properties of the curvature tensor and recurrence vector fields in it. In the present paper author decomposes the curvature tensor fields $\bar{K}^i_{jkh}$ (Art. 1) and K^i_{jkh}(Art. 2 & 3) in the recurrent generalised Finsler spaces (RGFn). He deals with the important properties of decomposition tensor fields and the recurrence vector field in RGFn. It is noted here that if the skew symmetric parts of the metric tensor field is taken to be zero, that is, if the skew symmetric parts of connection parameters P^{*i}_{jk} and Δ^i_{jk} is zero, the results obtained in RGFn being similar to that of recurrent Finsler spaces [3].

0. Preliminaries

Let us consider an n-dimensional Finsler space Fn endowed with a local coordinate system x^i [2] in which distance function $F(x, dx)$ satisfies the following properties

(i) $F(x, dx)$ is continuously differentiable at least four times in its $2n$ arguments.

(ii) $F(x, dx)$ is positive provided all dx^i are not zero.

(iii) $F(x, dx)$ is positively homogeneous of the first degree in dx^i.

(iv) $\dot{\partial}^{\cdot 2}_{ij} F^2(x, \dot{x})\ \xi^i \xi^j$ [3] >0 with $\sum_i (\xi^i)^2 \neq 0$ for any given $\dot{x}^i$.

The metric tensor $g_{ij}\ (x, \dot{x})$ of F_n is considered here as non-symmetric in general. The round and square brackets will be used to denote its symmetric and skew-symmetric parts respectively. For example

$$g_{(ij)} = \frac{1}{2}(g_{ij} + g_{ji})$$

and

$$g_{[ij]} = \frac{1}{2}(g_{ij} - g_{ji}).$$

1) Numbers in brackets refer to the referrence at the end of the paper
2) Indices $i, j, k, \cdots\cdots$ always take values from $1, 2, \cdots;\ n$.
3) $\dot{\partial}_i = \partial/\partial \dot{x}^i$.

The conjugate tensor of $g_{(ij)}$ is represented by g^{ij} and hence $g_{(ij)}\, g^{jk}=\delta^k_i$. The space endowed with this metric tensor are known as generalised Finsler spaces and we denote them by GFn.

The connection parameters for the locally Minkowskian and locally Euclidean GFn are denoted by P^{*i}_{jk} and Γ^{*i}_{jk} respectively. Let X^i be a vector field of GFn then the two processes of differentiation are defined as under.

(0.1) $$X^i_{,j}=\partial_j^{4)} X^i+\partial_j\dot{x}^h\dot{\partial}_h X^i+P^{*i}_{kj}X^k$$

and

(0.2) $$X^i|_j=\partial_j X^i-\Gamma^h_{kj}\,\dot{\partial}_h X^i\frac{\dot{x}^k}{F}+\Gamma^{*i}_{kj}X^k,$$

where

$$\Gamma^i_{jk}=\Gamma^{*i}_{jk}+C^i_{jh}\Gamma^{*h}_{rk}\dot{x}^r$$

and

$$C_{ijk}=\frac{1}{4}\dot{\partial}^3_{ijk}F^2(x,\dot{x}).$$

The commutation formulae involving the curvatue tensor fields are given by [1]

(0.3) $$2X^i_{,[jk]}=X^h\bar{K}^i_{hkj}-2X^i_{,h}\Delta^h_{[jk]}$$

and

(0.4) $$2X^i|_{[jk]}=\dot{\partial}_h X^i K^h_{ojk}F+X^h K^i_{hkj}-2X^i|_h\Delta^h_{[jk]},$$

where

$$\Gamma^{*i}_{[jk]}=P^{*i}_{[jk]}=\Delta^i_{[jk]}$$

and

(0.5) $$K^i_{okh}=K^i_{jkh}l^j,$$

We also have

(0.6) $$\dot{\partial}_k\Gamma^{*i}_{jk}\dot{x}^j\dot{x}^k=0.$$

The unit vector field l^j satisfies the relation

(0.7) $$l^j=\frac{\dot{x}^j}{F},\quad l^j|_k=0.$$

We have noted that

(0.8) $$F|_j=0.$$

The identites satisfied by the curvature tensor fields of GFn are stated below:

(0.9) $$\bar{K}^i_{jkh}+\bar{K}^i_{khj}+\bar{K}^i_{hjk}=2\Delta_{[j|k|h]\,;\,l}\,g^{il},$$

where (;) denotes covariant derivative based upon the connection parameter given by $Q^*_{jkh}=P^*_{jkh}+g_{(jk),h}$

4) $\partial_j=\partial/\partial x^j$.

(0.10) $$K^i_{jkh}+K^i_{khj}+K^i_{hjk}=2\Delta_{[j|k|h]il}g^{il},$$

where (i) denotes covariant derivative based upon the connection parameter given by $R^*_{jkh}=\Gamma^*_{jkh}$,

(0.11) $$\bar{K}^i_{jkh,l}+\bar{K}^i_{jhl,k}+\bar{K}^i_{jlk,h}+2[\bar{K}^i_{jmk}P^{*m}_{[lh]} +\bar{K}^i_{jmh}P^{*m}_{[lk]}+\bar{K}^i_{jml}P^{*m}_{[hk]}]=0,$$

(0.12) $$K^i_{jkh}|_l+K^i_{jhl}|_k+K^i_{jlk}|_h+F(K^m_{ohk}\dot{\partial}_m\Gamma^{*i}_{jl} +K^m_{ohl}\dot{\partial}_m\Gamma^{*i}_{jk}+K^m_{olk}\dot{\partial}_m\Gamma^{*i}_{hj})=2(K^i_{jml}\Delta^m_{[kh]} +K^i_{jmk}\Delta^m_{[hl]}+K^i_{jmk}\Delta^m_{[lk]}$$

and

(0.13) $$\bar{K}^i_{jkh}=-\bar{K}^i_{jhk},\quad K^i_{jkh}=-K^i_{jhk}.$$

Sinha and Singh [2] have defined recurrent curvature tensor field in GFn as follows:

The GFn, in which there exists a non-zero vector v_l such that the curvature tensor fields $\bar{K}^i_{jkh}$ and K^i_{jkh} satisfy the relations

(0.14) $$\bar{K}^i_{jkh,l}=v_l\bar{K}^i_{jkh}$$

and

(0.15) $$K^i_{jkh}|_l=v_lK^i_{jkh}$$

respectively, are said to be recurrent GFn (RGFn in short) and the curvature tensor fields of these spaces are called recurrent curvature tensor fields. Here v_l is known as recurrence vector field.

1. Decomposition of curvature tensor field $\bar{K}^i_{jkh}$

Let us consider the decomposition of curvature tensor field $\bar{K}^i_{jkh}$ of RGFn in the following form

(1.1) $$\bar{K}^i_{jkh}=X^i\alpha_{jkh},$$

where α_{jkh} is decomposition tensor field and X^i is a vector field such that

(1.2) $$X^iv_i=1.$$

From(1.1), the equation (0.13) yields

(1.3) $$\alpha_{jkh}=-\alpha_{jhk}.$$

We further decompose the tensor field α_{jkh} as under

(1.4) $$\alpha_{jkh}=v_j\alpha_{kh},$$

which implies

(1.4) a $$\alpha_{kh}=-\alpha_{hk}$$

in view of (1.3).

The equation (0.9) can be written as

(1.5) $$X^i[\alpha_{jkh}+\alpha_{khj}+\alpha_{hjk}]=2\Delta_{[j|k|h];l}\,g^{il}.$$

Transvecting (1.5) by v_i, we get

(1.6) $$\alpha_{jkh}+\alpha_{khj}+\alpha_{hjk}=2\Delta_{[j|k|h];\,l}v^l$$

with the help of (1.2).

Thus accordingly we have

THEOREM 1.1. *In RGFn, the decomposition tensor field satisfies the identity*
$$\alpha_{jkh}+\alpha_{khj}+\alpha_{hjk}=2\Delta_{[j|k|h]\,;\,l}v^l\ .$$

Under the decomposition (1.1), the Bianchi identity (0.11) takes the form

(1.7) $$[v_l\alpha_{jkh}+v_k\alpha_{jhl}+v_h\alpha_{jlk}]+2[\alpha_{jmk}\overset{*}{P}{}^m_{[lh]}+\alpha_{jmh}\overset{*}{P}{}^m_{[kl]}+\alpha_{jml}\overset{*}{P}{}^m_{[hk]}]=0$$

with the help of (0.14).

Now under the decomposition (1.4), the equation (1.7) reduces to

(1.8) $$[\alpha_{lkh}+\alpha_{khl}+\alpha_{hlk}]+2[\alpha_{mk}\overset{*}{P}{}^m_{[lh]}+\alpha_{mh}\overset{*}{P}{}^m_{[kl]}+\alpha_{ml}\overset{*}{P}{}^m_{[hk]}]=0.$$

From (1.6), the equation (1.8) can be written as

(1.9) $$\Delta_{[l|k|h]\,:\,m}v^m+\alpha_{mk}\overset{*}{P}{}^m_{[lh]}+\alpha_{mh}\overset{*}{P}{}^m_{[kl]}+\alpha_{ml}\overset{*}{P}{}^m_{[hk]}=0.$$

By virtue of (1.4)a, the equation (1.9) yields

(1.10) $$\alpha_{km}\overset{*}{P}{}^m_{[lk]}+\alpha_{hm}\overset{*}{P}{}^m_{[kl]}+\alpha_{lm}\overset{*}{P}{}^m_{[hk]}=\Delta_{[l|k|h];\,m}v^m\ .$$

Hence we have

THEOREM 1.2. *In RGFn, the decomposition tensor field satisfies the following identity*
$$\alpha_{km}\overset{*}{P}{}^m_{[lh]}+\alpha_{hm}\overset{*}{P}{}^m_{[kl]}+\alpha_{lm}\overset{*}{P}{}^m_{[hk]}=\Delta_{[l|k|h];\,m}v^m.$$

By virtue of (1.3) and (1.4), the equation (1.6) reduces to

(1.11) $$\alpha_{jkh}+\alpha_{hj}v_k-v_h\alpha_{kj}=2\Delta_{[j|k|h];\,l}v^l$$

which can be written as

(1.12) $$\alpha_{jkh}=2[v_{[h}\alpha_{k]j}+\Delta_{[j|k|h];\,l}v^l].$$

Transvecting (1.12) by X^i and using (1.1), we get

(1.13) $$\bar{K}^i_{jkh}=2X^i[v_{[h}\alpha_{k]j}+\Delta_{[j|k|k];\,l}v^l].$$

Thus we can write

THEOREM 1.3. *In RGFn, the curvature tensor field* $\bar{K}^i_{jkh}$ *can also be expressed* *in terms of decomposition tensor field* α_{jk} *as under*

$$\bar{K}^i_{jkh}=2X^i[v_{[h}\alpha_{k]j}+\Delta_{[j|k|h];\,l}v^l].$$

Differentiating (1.1) covariantly and noting (0.14), we obtain

(1.14) $$v_l\bar{K}^i_{jkh}=X^i_{,l}\alpha_{jkh}+X^i\alpha_{jkh,l}\ .$$

From (1.1), it reduces to

(1.15) $$X^i(v_l\alpha_{jkh}-\alpha_{jkh,l})=X^i_{,l}\,\alpha_{jkh}.$$

If we consider X^i to be covariant constant, the equation (1.15) yields

(1.16) $$X^i(v_l\alpha_{jkh}-\alpha_{jkh,l})=0.$$

Since X^i is an arbitrary vector field, hence we have

(1.17) $$\alpha_{jkh,l}=v_l\alpha_{jkh}.$$

Conversely if (1.17) is true, the equation (1.14) takes the form

(1.18) $$v_l\bar{K}^i_{jkh}=X^i_{,l}\,\alpha_{jkh}+X^iv_l\alpha_{jkh}.$$

By means of (1.1) it reduces to

(1.19) $$X^i_{,l}\,\alpha_{jkh}=0.$$

Since $\alpha_{jkh}\neq 0$, hence we get

(1.20) $$X^i_{,l}=0,$$

which implies X^i is covariant constant.

Accordingly we have

THEOREM 1.4. *In RGFn, the necessary and sufficient condition for the decomposition tensor field* α_{jkh} *to be recurrent is that the vector field* X^i *is covariant constant.*

Now, taking covariant differentiation of (1.2) we get

(1.21) $$X^i_{,l}\,v_i+X^iv_{i,l}=0.$$

From (1.21) we conclude

COROLLARY 1.1. *In RGFn, If* X^i *is covariant constant it implies that the recurrence vector field* v_i *is also covariant constant.*

Considering covariant differentiation of (1.4) and using (1.17), we obtain

(1.22) $$v_l\alpha_{jkh}=v_{j,l}\alpha_{kh}+v_j\alpha_{kh,l}\ .$$

From (1.4) and Cor. 1.1, the equation (1.22) becomes

(1.23) $$v_jv_l\alpha_{kh}=v_j\alpha_{kh,l}\,.$$

Since $v_j\neq 0$, we have

(1.24) $$\alpha_{kh,l}=v_l\alpha_{kh}\,.$$

Hence we write

THEOREM 1.5. *In RGFn, if the decomposition tensor field α_{jkh} is recurrent, then the tensor field α_{kh} is recurrent and the converse is also true under the assumption that the vector X^i is covariant constant.*

In view of (1.17) and (1.4) the equation (1.7) becomes

(1.25) $\alpha_{jkh,l}+\alpha_{jhl,k}+\alpha_{jlk,h}=2(\alpha_{jkm}\overset{*}{P}{}^{m}_{[lk]}+\alpha_{jhm}\overset{*}{P}{}^{m}_{[kl]}+\alpha_{jlm}\overset{*}{P}{}^{m}_{[hk]}$.

Also by means of decomposition (1.4), the equation (1.7) gives

(1.26) $v_j\,[v_l\alpha_{kh}+v_k\alpha_{hl}+v_h\alpha_{lk}+2(\alpha_{mk}\overset{*}{P}{}^{m}_{[lh]}+\alpha_{mh}\overset{*}{P}{}^{m}_{[kl]}+\alpha_{ml}\overset{*}{P}{}^{m}_{[hk]})]=0.$

Transvecting (1.26) by X^j and making use of (1.2), (1.4)a and (1.24), we obtain

(1.27) $\alpha_{kh,l}+\alpha_{hl,k}+\alpha_{lk,h}=2\,[\alpha_{km}\overset{*}{P}{}^{m}_{[lh]}+\alpha_{hm}\overset{*}{P}{}^{m}_{[kl]}+\alpha_{lm}\overset{*}{P}{}^{m}_{[hk]}]$.

Thus we have

THEOREM 1.6. *In RGFn, the decomposition tensor fields α_{jkh} and α_{kh} satisfy the Bianchi identities*

$$\alpha_{jkh,l}+\alpha_{jhl,k}+\alpha_{jlk,h}=2\,[\overset{*}{P}{}^{m}_{[lh]}\,\alpha_{jkm}+\overset{*}{P}{}^{m}_{[kl]}\alpha_{jhm}+\alpha_{jlm}\overset{*}{P}{}^{m}_{[hk]}]$$

and

$$\alpha_{kh,l}+\alpha_{hl,k}+\alpha_{lk,h}=2\,[\overset{*}{P}{}^{m}_{[lh]}\alpha_{km}+\overset{*}{P}{}^{m}_{[kl]}\alpha_{hm}+\overset{*}{P}{}^{m}_{[hk]}\alpha_{lm}]$$

respectively with the condition that X^i is covariant constant.

Taking covariant differentiation of (0.14) and commuting the indices l and m, we get

(1.28) $$\bar{K}^i_{jkh,lm}-\bar{K}^i_{jkh,ml}=(v_{l,m}-v_{m,l})\bar{K}^i_{jkh}$$

With help of commutation formula (0.3) it takes the form

(1.29) $\bar{K}^r_{jkh}\bar{K}^i_{rlm}-\bar{K}^i_{rkh}\bar{K}^r_{jlm}-\bar{K}^i_{jrm}\bar{K}^r_{klm}-\bar{K}^i_{jkr}\bar{K}^r_{hlm}-2\bar{K}^i_{jkh,r}\Delta^r_{[lm]}=(v_{l,m}-v_{m,l})\bar{K}^i_{jkh}$

From (1.1), (1.2) and (1.4), the equation (1.29) becomes

(1.30) $-X^iv_jX^rv_k\alpha_{rh}\alpha_{lm}-X^iv_hX^rv_j\alpha_{lm}\alpha_{kr}-2X\,v_jv_r\Delta^r_{[lm]}\alpha_{kh}=X^iv_j(v_{l,m}-v_{m,l})\alpha_{kh}.$

Transvecting (1.30) by X^jv_i and noting (1.4) a, we have

(1.31) $$(v_k\alpha_{hr}-v_h\alpha_{kr})X^r\alpha_{lm}-2\alpha_{kh}v_r\Delta^r_{[lm]}=(v_{l,m}-v_{m,l})\alpha_{kh}.$$

Now if $\alpha_{kr}X^r=0$, then the equation (1.31) gives

(1.32) $$\alpha_{kh}v_r\Delta^r_{[lm]}=v_{[m,l]}\alpha_{kh}.$$

But α_{kh} is arbitrary tensor field, therefore the above equation reduces to

(1.33) $$v_r\Delta^r_{[lm]}=v_{[m,l]}.$$

Conversely if, the equation (1.33) is true, equation (1.30) yields

(1.34) $$X^i X^r v_j(\alpha_{rh} v_k \alpha_{lm} + \alpha_{kr} v_h \alpha_{lm}) = 0.$$

Multiplying the equation (1.34) by $X^j v_i$ and noting (1.2) and (1.4) a we obtain

(1.35) $$X^r \alpha_{rh} v_k = \alpha_{rk} v_h \lambda X^r.$$

Transvecting the equation (1.35) by $X^h X^k$, we get

(1.36) $$\alpha_{rh} X^h = \alpha_{rk} X^k,$$

which implies

$$\alpha_{rh} X^h = 0.$$

Hence we have

THEOREM 1.7. *In RGFn, the necessary and sufficient condition for the relation*

$$v_r \Delta^r_{[lm]} = v_{[m,l]}$$

to be true is that

$$\alpha_{kr} X^r = 0.$$

2. Decomposition of curvature tensor field

In RGFn, we decompose the curvature tensor field K^i_{jkh} as under

(2.1) $$K^i_{jkh} = \dot{x}^i \beta_{jkh},$$

where β_{jkh} is a homogeneous decomposition tensor field of degree -1 in $\dot{x}^i$.
From (0.13) and (2.1), we have

(2.2) $$\beta_{jkh} = -\beta_{jhk}.$$

Under the decomposition (2.1), the equation (0.5) yields

(2.3) $$\beta_{okh} = \beta_{jkh} l^j$$

hence we can write

(2.4) $$K^i_{okh} = \beta_{okh} \dot{x}^i.$$

In the equation (2.4) contracting the indices i and h, we obtain

(2.5) $$K_{ok} = \beta_{ok},$$

where

$$\beta_{ok} = \beta_{okh} \dot{x}^h.$$

We also have

(2.6) $$\beta_{okh} = -\beta_{ohk}.$$

Now contracting the indices i and h in the equation (2.1), we get

(2.7) $$K_{jk} = \beta_{jk}$$

where

(2.8) $$\beta_{jkh}\dot{x}^h=\beta_{jk}.$$

By virtue of decomposition (2.1), the equation (0.10) gives

(2.9) $$\dot{x}^i(\beta_{jkh}+\beta_{khj}+\beta_{hjk})=2\Delta_{[j|k|h]il}g^{il}.$$

Contracting the indices i,h and using (0.13), (2.3) and (2.8) in the equation (2.9), it yields

(2.10) $$\beta_{ojk}=\frac{2}{F}[\beta_{[kj]}+\Delta_{[j|k|i]il}g^{il}].$$

Accordingly we have

THEOREM 2.1. *In RGFn, the decomposition tensor field β_{ojk} can be expressed in the form*

$$\beta_{ojk}=\frac{2}{F}[\beta_{[kj]}+\Delta_{[j|k|i]il}g^{il}].$$

Considering covariant differentiation of (2.1) and noting (0.7) and (0.15), we obtain

(2.11) $$v_lK^i_{jkh}=\dot{x}^i\beta_{jkh}|_l$$

which yields

(2.12) $$v_l\beta_{jkh}=\beta_{jkh}|_l.$$

Transvecting the equation (2.12) by l^j and simplifying the result by means of (0.7) and (2.3), we get

(2.13) $$v_l\beta_{okh}=\beta_{okh}|_l.$$

Also contracting the indices i,h in the equation (2.11) and using (0.7), (2.1) and (2.8) it gives

(2.14) $$v_l\beta_{jk}=\beta_{jk}|_l.$$

From the equation (2.13), we also have

$$v_l\beta_{ok}=\partial_l\beta_{ok}|_l.$$

Thus we have

THEOREM 2.2. *In RGFn, the decomposition tensor fields β_{jkh}, β_{okh}, β_{jk} and β_{ok} behave like recurrent tensor fields.*

Differentiating (2.14) covariantly with respect to the index m and commuting the indices l,m, we have

(2.15) $$(v_l|_m-v_m|_l)\beta_{jk}=2\beta_{jk}|_{[lm]}.$$

By virtue of (0.4), (2.1), (2.4), (2.8) and (2.14), the equation (2.15) reduces to

(2.16) $$(v_l|_m-v_m|_l)\beta_{jk}=\beta_{jk}FB_{olm}-\beta_{pk}\dot{x}^p\beta_{jlm}-\beta_{jp}\dot{x}^p\beta_{klm}-2v_p\beta_{jk}\Delta^p{}_{[lm]}.$$

Transvectng (2.16) by l^j and noting (0.7) iand (2.3), we obtain

(2.17) $$2[v_{[l|m]}+v_p\Delta^p_{[lm]}]\beta_{ok}=-\beta_{op}\dot{x}^p\beta_{klm},$$

where $\beta_{ok}=\beta_{jk}l^j$. If we suppose that $\beta_{op}\dot{x}^p=0$, the equation (2.17) takes the following form

(2.18) $$[v_{[l|m]}+v_p\Delta^p_{[lm]}]=0,$$

since $\beta_{ok}\neq 0$.

Conversely if (2.18) is true, the equation (2.17) reduces to

(2.19) $$\beta_{op}\dot{x}^p\beta_{klm}=0.$$

But $\beta_{klm}\neq 0$, therefore we have

(2.20) $$\beta_{op}\dot{x}^p=0.$$

Hence we have

THEOREM 2.3. *In RGFn, the necessary and sufficient condition for the relation*

$$v_p\Delta^p_{[lm]}=v_{[m|l]}$$

to be true is that

$$\beta_{op}\dot{x}^p=0.$$

By means of (0.15), (2.1) and (2.4), the Bianchi identity (0.12) takes the form

(2.21) $$\dot{x}^i v_l\beta_{jkh}+\dot{x}^i v_k\beta_{jhl}+\dot{x}^i v_h\beta_{jlk}+\dot{x}^m F(\beta_{ohk}\dot{\partial}_m\Gamma^{*i}_{jl}+\beta_{ohl}\dot{\partial}_m\Gamma^{*l}_{jk}+\beta_{olk}\dot{\partial}_m\Gamma^{*i}_{hj})$$
$$=2\dot{x}^i[\beta_{jml}\Delta^m_{[kh]}+\beta_{jmk}\Delta^m_{[hl]}+\beta_{jmh}\Delta^m_{[lk]}].$$

Transvecting (2.21) by l^j and simplifying by virtue of (0.6) and (2.3), we have

(2.22) $$v_l\beta_{okh}+v_k\beta_{ohl}+v_h\beta_{olk}=2[\beta_{oml}\Delta^m_{[kh]}+\beta_{omk}\Delta^m_{[hl]}+\beta_{omh}\Delta^m_{[lk]}].$$

In view of (2.13), it becomes

(2.23) $$\beta_{okh|l}+\beta_{ohl|k}+\beta_{olk|h}=2[\beta_{oml}\Delta^m_{[kh]}+\beta_{omk}\Delta^m_{[hl]}+\beta_{omh}\Delta^m_{[lk]}].$$

Accordingly we write

THFOREM 2.4. *In RGFn, the decomposition tensor field satisfies the following Bianchi identity*

$$\beta_{okh|l}+\beta_{ohl|k}+\beta_{olk|h}=2[\beta_{oml}\Delta^m_{[kh]}+\beta_{omk}\Delta^m_{[hl]}+\beta_{omh}\Delta^m_{[lk]}].$$

3. Another decomposition of curvature tensor field K^i_{jkh}

In this section the curvature tensor field K^i_{jkh} is decomposed in the following manner

(3.1) $$K^i_{jkh}=X^i\theta_{jkh},$$

where θ_{jkh} is a suitable decomposition tensor field and X^i is a vector field such that $X^i v_i=1$, where v_i is recurrence vector field. Interchanging the indices k, h in (3.1) and using (0.13), we get

(3.2) $$\theta_{jkh}=-\theta_{jhk}.$$

Transvecting (3.1) by l^j and noting (0.5), we have

(3.3) $$\theta_{okh}=\theta_{jkh}l^j,$$

where

(3.4) $$K^i_{okh}=X^i\theta_{okh}.$$

The decomposition tensor field θ_{okh} satisfies the identity

(3.5) $$\theta_{okh}=-\theta_{ohk},$$

in view of (0.13).

The identity (0.10) can be written in the form

(3.6) $$X^i(\theta_{jkh}+\theta_{khj}+\theta_{hjk})=2\Delta_{[j|k|h]il}g^{il},$$

With help of (3.1)

Transvecting the equation (3.6) by v_i and noting (1.2), we get

(3.7) $$\theta_{jkh}+\theta_{khj}+\theta_{hjk}=2\Delta_{[j|k|h]il}v^l.$$

Thus we have

THEOREM 3.1. *In RGFn, the decomposition tensor field θ_{jkh} satisfies the identity*

$$\theta_{jkh}+\theta_{khj}+\theta_{hjk}=2\Delta_{[j|k|h]il}v^l.$$

Transvecting (3.7) by l^j and simplifying the result by means of (3.2) and(3.3), we obtain

(3.8) $$\theta_{okh}=2\theta_{[hk]o}+2\Delta_{[j|k|h]il}v^l l^j,$$

where

$$\theta_{[hk]j}l^j=\theta_{[hk]o}.$$

Multiplying the equation (3.8) by X^i and using (3.4), we have

(3.9) $$K^i_{okh}=2X^i[\theta_{[hk]o}+l^j\Delta_{[j|k|h]il}v^l].$$

We have

THEOREM 3.2. *In RGFn, the curvature tensor field K^i_{okh} can be expressed in terms of the decomposition θ_{hko} as under:*

$$K^i_{okh}=2X^i[\theta_{[hk]o}+l^j\Delta_{[j|k|h]il}v^l].$$

Taking covariant differentiation of (3.1) and making use of (0.15), we obtain

(3.10) $$v_l K^i_{jkh} = X^i|_l \theta_{jkh} + x^i \theta_{jkh|l}.$$

Considering X^i to be covariant constant and noting (3.1), the equation (3.10) gives

(3.11) $$v_l \theta_{jkh} = \theta_{jkh}|_l.$$

Transvecting (3.11) by l^j and taking into consideration (0.7) and (3.3), we get

(3.12) $$v_l \theta_{okh} = \theta_{okh}|_l.$$

Conversely if (3.11) is true, the equation (3.10) reduces to

(3.13) $$v_l K^i_{jkh} = X^i|_l \theta_{jkh} + X^i v_i \theta_{jkh}.$$

By virtue of (3.1), the equation (3.13) yields

(3.14) $$\theta_{jkh} X^i|_l = 0.$$

Since $\theta_{jkh} \neq 0$, therefore we have

(3.15) $$X^i|_l = 0,$$

that is X^i is covariant constant.

Hence we conclude

THEOREM 3.3. *In RGFn, the necessary and sufficient condition for the decomposition tensor fields θ_{jkh} and θ_{okh} to be recurrent is that the vector field X^i is covariant constant.*

In view of (3.1), (0.15) and (3.4), the Bianchi identity (0.12) becomes

(3.16) $$X^i [v_l \theta_{jkh} + v_k \theta_{jhl} + v_h \theta_{jlk}] + F X^m [\theta_{ohk} \dot{\partial}_m \Gamma^{*i}_{jl} + \theta_{ohl} \dot{\partial}_m \Gamma^{*i}_{jk} + \theta_{olk} \dot{\partial}_m \Gamma^{*i}_{hj}]$$
$$= 2X^i [\theta_{jmk} \Delta^m_{[hl]} + \theta_{jml} \Delta^m_{[kh]} + \theta_{jmh} \Delta^m_{[lk]}].$$

Transvecting the equation (3.16) by l^j it gives

(3.17) $$v_l \theta_{okh} + v_k \theta_{ohl} + v_h \theta_{olk} = 2[\theta_{omk} \Delta^m_{|hl|} + \theta_{oml} \Delta^m_{[kh]} + \theta_{omh} \Delta^m_{[lk]}]$$

by means of (0.6) and (3.3).

Now, under the assumption that X^i is covariant constant, the equation (3.17) reduces to

(3.18) $$\theta_{okh}|_l + \theta_{ohl}|_k + \theta_{olk}|_h = 2[\theta_{omk} \Delta^m_{[hl]} + \theta_{oml} \Delta^m_{[kh]} + \theta_{omh} \Delta^m_{[lk]}],$$

which is Bianchi identity for the decomposition tensor field θ_{okh}. Accordingly we have,

THEOREM 3.4. *In RGFn, the decomposition tensor field satisfies the Bianchi identity*

$$\theta_{okh}|_l + \theta_{ohl}|_k + \theta_{olk}|_h = 2[\theta_{omk} \Delta^m_{[hl]} + \theta_{oml} \Delta^m_{[kh]} + \theta_{omh} \Delta^m_{[lk]}],$$

under the assumption that X^i is covariant constant, which is a necessary and sufficient condition for θ_{okh} to be recurrent.

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