Kyungpook Math. J.
Volume 15, Number 2
December, 1975

PROPERTIES OF c-CONTINUOUS AND c^*-CONTINUOUS FUNCTIONS

By Paul E. Long and Larry L. Herrington

1. Introduction

In this paper we further the investigations of c-continuous functions found in [3] and [5] and of c^*-continuous functions found in [7]. Professors Gentry and Hoyle [3] defined the concept of c-continuous functions as follows:

DEFINITION 1.1. A function $f:X\to Y$ is *c-continuous* if for each $x\in X$ and each open $V\subset Y$ containing $f(x)$ and having compact complement, there exists an open U containing x such that $f(U)\subset V$.

Two useful characterizations of c-continuous functions are contained in the next theorem.

THEOREM 1.2. *Let $f:X\to Y$ be a function. Then f is c-continous if and only if*
(*a*) [3] *The inverse image of every open subset of Y having compact complement is open in X.*
(*b*) [5] *The inverse image of every closed compact subset of Y is closed in X.*

Professor Park [7] defined the concept of c^*-continuous functions in the following manner:

DEFINITION 1.3. The function $f:X\to Y$ is *c^*-continuous* if for each countably compact and closed $C\subset Y$, $f^{-1}(C)$ is closed in X. Equivalently, if $V\subset Y$ is open and has countably compact complement, then $f^{-1}(V)$ is open in X.

As noted in [7], every function that is c^*-continuous is also c-continuous but the converse need not hold. Of course, if Y is a space where the concepts of compactness and countable compactness agree, then a c-continuous function $f:X\to Y$ would also be c^*-continuous. Paracompact spaces, Lindelof spaces and metric spaces are examples of such spaces.

We denote the closure of a set A as $\mathrm{cl}(A)$ and the interior by $\mathrm{Int}(A)$.

2. Properties of c-continuous functions

In this section we continue the investigations of [3] and [5] concerning c-continuous functions.

THEOREM 2.1. *For each $\alpha \in A$, let Y_α be a locally compact Hausdorff space and let $f_\alpha : X \to Y$ be c-continuous. Then the function $f : X \to \prod_\alpha \{Y_\alpha : \alpha \in A\}$ defined by $f(x) = \{f_\alpha(x)\}$ is c-continuous.*

PROOF. We first show the graph of f is closed. To do this, let $x_0 \in X$ and let $\{y_\alpha^0\} \in \prod_\alpha Y_\alpha$ be different from $f(x_0) = \{f_\alpha(x_0)\}$. Then there exists a $\beta \in A$ such that $f_\beta(x_0) \neq y_\beta^0$. Since the graph of each f_α is closed [5, Theorem 8], there exist open sets U and V containing x_0 and y_β^0, respectively, such that $f_\beta(U) \cap V = \phi$ by the Lemma of [6]. Therefore, $f(U) \cap (V \times \prod_{\alpha \neq \beta} Y_\alpha) = \phi$. Using the Lemma of [6] again, we conclude the graph of f is closed. Now Theorem 7 of [5] gives that f is c-continuous.

COROLLARY. *If $f : X \to Y$ is c-continuous, where both X and Y are locally compact Hausdorff spaces, then the graph function $g : X \to X \times Y$ defined by $g(x) = (x, f(x))$ is c-continuous.*

The Corollary gives a somewhat improved version of Theorem 11 of [5].

The converse of Theorem 2.1 does not hold as the following example shows.

EXAMPLE 2.2. Let R be the reals with the usual topology and $I = [0, 1]$ have the subspace topology. Define $f_1 : I \to R$ and $f_2 : I \to R$ as

$$f_1(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } 0 < x \leq 1 \end{cases}$$

and

$$f_2(x) = \begin{cases} 1 & \text{if } x = 0 \\ \dfrac{1}{x} & \text{if } 0 < x \leq 1. \end{cases}$$

Now define $f : I \to R \times R$ by $f(x) = (f_1(x), f_2(x))$. Then it is easily seen that f is c-continuous, but f_1 is not c-continuous.

THEOREM 2.3. *For each $\alpha \in A$, let $f_\alpha : X_\alpha \to Y_\alpha$ be a function and let Y_α be locally compact. Define $f : \prod_\alpha X_\alpha \to \prod_\alpha Y_\alpha$ as $f(\{x_\alpha\}) = \{f_\alpha(x_\alpha)\}$. If f is c-continuous then each f_α is c-continuous.*

PROOF. Let K_β be a closed compact subset of Y_β and let $\{y_\alpha^0\}$ be a point in the range of f. Then, since each Y_α is locally compact, there exists an open U_α containing y_α^0 such that $\mathrm{cl}(U_\alpha)$ is compact. Hence, $K_\beta \times \prod_{\alpha \neq \beta} \mathrm{cl}(U_\alpha)$ is a closed compact subset of $\prod_\alpha Y_\alpha$. Therefore, $f_\beta^{-1}(K_\beta \times \prod_{\alpha \neq \beta} \mathrm{cl}(U_\alpha)) = f^{-1}(K_\beta) \times \prod_{\alpha \neq \beta} f_\alpha^{-1} \mathrm{cl}(U_\alpha)$

is closed in $\prod_{\alpha} X_\alpha$ by Theorem 2 of [5]. It follows that $f_\beta^{-1}(K_\beta)$ is closed in X_β so that f_β is c-continuous, again by Theorem 2 of [5].

The boundedness of a c-continuous function plays an important role in determining its continuity as shown by our next theorem.

THEOREM 2.4. *Let $f:X\to Y$ be a c-continuous function from a space X into a metric space Y which has the property that closed bounded sets are compact. If f is bounded on an open $U\subset X$, then f is continuous at every point of U.*

PROOF. By Theorem 2 of [3], $f|U:U\to Y$ is c-continuous, and, since f is bounded on U, $f(U)$ lies in a closed bounded, hence compact, subset of Y. Theorem 5 of [3] then gives $f|U$ continuous.

This theorem tells us, for instance, that if $f:X\to R^n$ is c-continuous and if for each point $x\in X$ there exists an open set U containing x such that f is bounded on U, then f is a continuous function.

Another theorem that has application in the reals follows Definition 2.5.

DEFINITION 2.5. [2] The space X has *property k_2* at $p\in X$ if for each subset A having p as an accumulation point, there is a subset B of A and a compact set $K\supset B\cup\{p\}$ such that p is an accumulation point of B. The space X is a *k_2-space* if it has property k_2 at each of its points.

THEOREM 2.6. *Let $f:X\to Y$ be c-continuous where X and Y are Hausdorff and X has property k_2 at $p\in X$. If f is compact preserving, then f is continuous at p.*

PROOF. The function f has closed point inverses by Theorem 2 of [5] and the continuity then follows from Theorem 4.4 of [2].

THEOREM 2.7. *Let X be regular and locally compact and let Y be locally compact and Hausdorff. Then $f:X\to Y$ is c-continuous if and only if one of the following conditions holds:*

(a) *f has a closed graph*

(b) *f is locally closed* [2] *and has closed point inverses.*

(c) *f maps compact sets onto closed sets and has closed point inverses.*

PROOF. By Theorems 7 and 8 of [5], f is c-continuous if and only if f has a closed graph. Theorem 3.11 of [2] then gives the conclusion.

DEFINITION 2.8. [8] Let $f:X\to Y$ be a function and $p\in Y$. Then the set of limit points of f at p, denoted by $L(f;p)$ is the set of all points $y \in Y$ such that there exists a sequence (x_n) in X converging to p for which $(f(x_n)) \to y$.

THEOREM 2.9. *If $f:X \to Y$ is c-continuous and Y is Hausdorff, then $L(f;p)=\{f(p)\}$.*

PROOF. Suppose $y_0 \in L(f;p)$ where $y_0 \neq f(p)$. Then there exists a sequence $(x_n) \to p$ for which $(f(x_n)) \to y_0$. Since Y is Hausdorff, there exist open disjoint sets U and V containing y_0 and $f(p)$, respectively. Also, there exists an $n_0 \in N$ such that if $n \geq n_0$, then $f(x_n) \in U$. Thus $\{f(x_n) | n \geq n_0\} \cup \{y_0\}$ is a closed compact set whose inverse under f is not closed because it does not contain p. This contradiction to Theorem 2 of [5] shows $y_0 = f(p)$ so that $L(f;p)=\{f(p)\}$.

COROLLARY 2.10. *Let X be first countable, Y Hausdorff and $f: X \to Y$ c-continuous. Then f is continuous at $p \in X$ if and only if f has an at worst removable discontinuity at p.*

THEOREM 2.11. *Let $f:X \to Y$ be a c-continuous function from the first countable space X into the first countable countably compact Hausdorff space Y. Then f is continuous.*

PROOF. Suppose f is not continuous at $p \in X$. Then the first countability of X gives the existence of a sequence $(x_n) \to p$ such that $(f(x_n)) \not\to f(p)$. Thus, there exists an open set V containing $f(p)$ such that for any $n_0 \in N$, there is an $n \geq n_0$ such that $f(x_n) \notin V$. Consequently, there is a subsequence $(f(x_{n_i}))$ of $f(x_n))$ on $Y-V$ which accumulates because Y is countably compact. Call the point of accumulation b and note that $b \in Y-V$ so that $b \neq f(p)$. The first countability of Y now gives, by Theorem 6.2 (2) of [1, p. 217] a subsequence $(f(x_{n_{i_k}}))$ of $(f(x_{n_i}))$ which converges to b. Thus, $b \in L(f;p)$. But this contradicts Theorem 2.9 and we conclude that if $(x_n) \to p$, then $(f(x_n)) \to f(p)$ showing f continuous at p.

Theorem 2 of [3] shows that a c-continuous $f:X \to Y$ may be restricted to any subset A of X and the resulting function $f|A:A \to Y$ will also be c-continuous. However, c-continuous functions may not, in general, be restricted in the range. That is, if $f:X \to Y$ is c-continuous, then $f:X \to f(X) \subset Y$ need not be c-continuous as is shown by the next example.

EXAMPLE 2.12. Let Y be the set of reals $[0, \infty)$ and let σ be the topology on Y generated by sets of the form (r, ∞) for $r > 1$, together with the sets $\{1\}$ and

$[0,1)$ as a base. Note that Y is not Hausdorff and the only open sets with compact complements are Y, $[0,1)\cup(1,\infty)$, $(1,\infty)$ and $[1,\infty)$. Let $X=\{1,2,3\}$ with topology $T=\{\{1\},\{2,3\},\{1,2,3\},\phi\}$ and define $f:X\to Y$ by $f(x)=x$. The function f is c-continuous. Now give $f(X)=\{1,2,3\}\subset Y$ the subspace topology $\sigma_{f(X)}$ of Y. This topology is $\sigma_{f(X)}=\{\{1\},\{3\},\{2,3\},\{1,2,3\},\phi\}$. Observe that $\{3\}$ is open in $(f(X),\sigma_{f(X)})$ and has compact complement, but $f^{-1}(\{3\})=\{3\}$ is not open in (X,T) showing $f:X\to f(X)$ is not c-continuous.

The following theorems give conditions under which we may restrict to $f(X)$.

THEOREM 2.13. *Let $f:X\to Y$ be a c-continuous function from a space X into a Hausdorff space Y. Then $f:X\to f(X)\subset Y$ is c-continuous.*

PROOF. Let $U\subset f(X)$ be an open subset of the subspace $f(X)$ such that $f(X)-U$ is compact in the subspace $f(X)$. Then $f(X)-U$ is a compact subset of Y and, since Y is Hausdorff, $f(X)-U$ is closed in Y. Thus, $V=Y-(f(X)-U)$ is open in Y and has a compact complement. The c-continuity of f gives $f^{-1}(V)$ open in X. Now, noting that $f^{-1}(V)=f^{-1}(Y)-f^{-1}(f(X)-U)=X-(f^{-1}(f(X))-f^{-1}(U))=X-(X-f^{-1}(U))=f^{-1}(U)$, we conclude $f:X\to f(X)$ is c-continuous.

THEOREM 2.14. *Let $f:X\to Y$ be c-continuous and suppose $f(X)\subset Y$ is closed. Then $f:X\to f(X)$ is c-continuous.*

PROOF. Let $U\subset f(X)$ be open in the subspace $f(X)$ where $f(X)-U$ is compact in the subspace $f(X)$. Now $f(X)-U$ is also closed and compact in Y. Thus $V=Y-(f(X)-U)$ is an open set in Y having compact complement. Therefore, $f^{-1}(V)=f^{-1}(U)$ is open showing $f:X\to f(X)$ is continuous.

THEOREM 2.15. *Let $f:X\to Y$ be c-continuous and suppose the graph of f, $G=\{(x,f(x)):x\in X\}$, is closed in $X\times Y$. Then $f:X\to f(X)$ is c-continuous.*

PROOF. We first show the graph of $f:X\to f(X)$ is closed. Then by Theorem 7 of [5], $f:X\to f(X)$ is c-continuous.

Let $(x,y)\notin G$ where $(x,y)\in X\times f(X)\subset X\times Y$. Then there exist open sets $U\subset X$ and $V\subset Y$ containing x and y, respectively, such that $(U\times V)\cap G=\phi$ because G is closed in $X\times Y$. Let $W=V\cap f(X)$. Then W is open in the subspace $f(X)$, contains y, and $U\times W\subset U\times V$ which implies $(U\times W)\cap G=\phi$. Thus G is closed in $X\times f(X)$ showing $f:X\to f(X)$ has a closed graph.

3. C^*-continuous functions

We begin our considerations of c^*-continuous functions $f:X\to Y$ by some observations concerning the space Y. For any topological space (Y,σ), the collection of open sets having countably compact complements form a base for a new topology σ^* on Y. The reason is that if U and V are open and have countably compact complements, then their intersection has a countably compact complement as may be shown by use of the equality $Y-(U\cap V)=(Y-U)\cup(Y-V)$. Of course, $\sigma^*\subset\sigma$ and (Y,σ^*) is always a countably compact space. If we consider the following diagram for any function f and any spaces X and (Y,σ),

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & (Y,\sigma) \\ & \searrow^{f} & \downarrow i \\ & & (Y,\sigma^*) \end{array}$$

we see that $f:X\to(Y,\sigma)$ is c^*-continuous if and only if $f:X\to(Y,\sigma^*)$ is continuous. Also, $i:(Y,\sigma)\to(Y,\sigma^*)$ is continuous and i^{-1} is c^*-continuous.

These remarks lead us to observe that a c^*-continuous fundamental group may be defined analogous to the c-continuous fundamental group of [4] and an investigation similar to that one made by making heavy use of the above diagram and its implications.

The remarks also lead us to an immediate generalization of Theorem 1 of [7].

THEOREM 3.1. *Let X be a space and $\{A_\alpha:\alpha\in A\}$ a cover of X such that either* (a) *the sets A_α are all open*

or (b) *the sets are all closed and form a neighborhood finite family.*

If $f:X\to(Y,\sigma)$ is a function such that $f|A_\alpha$ is c^*-continuous, then f is c^*-continuous.

PROOF. Since each $f|A_\alpha:A_\alpha\to(Y,\sigma)$ is c^*-continuous, our remarks about the above diagram show that $f|A_\alpha:A_\alpha\to(Y,\sigma^*)$ is continuous. Theorem 9.4 of [1, Chapter 3, p. 83] then gives $f:X\to(Y,\sigma^*)$ continuous. Using the remarks following the diagram again, we see that $f:X\to(Y,\sigma)$ is c^*-continuous.

Since an analogous diagram holds for c-continuous functions, we note that Theorem 4 of [3] could be generalized in the same manner as Theorem 3.1.

Professor Park in [7] defines a space X to be locally countably compact if it is Hausdorff and each point has a relatively countably compact neighborhood, i.e., for each $x\in X$ there is an open $U\subset X$ containing x such that $\mathrm{cl}(U)$ is countably compact. We now show that the regularity condition on Y in Lemma 8 of [7] may be removed. Upon so doing, that Lemma may be stated as follows:

THEOREM 3.2. *Let $f:X\to Y$ be c^*-continuous and let Y be a locally countably compact space. Then $G(f)$ is closed.*

PROOF. Let $x\in X$ and let $y\in Y$ where $y\neq f(x)$. Since Y is Hausdorff, there is an open V containing y such that $f(x)\notin \mathrm{cl}(V)$. The local countable compactness of Y gives the existence of an open set W containing y such that $\mathrm{cl}(W)$ is countably compact. Thus, $V\cap W$ is an open set containing y and, since closed subsets of countably compact spaces are countably compact, $\mathrm{cl}(V\cap W)\subset \mathrm{cl}(V)\cap \mathrm{cl}(W)$ shows $\mathrm{cl}(V\cap W)$ is countably compact and furthermore does not contain $f(x)$. Therefore, $Y-\mathrm{cl}(V\cap W)$ is an open set containing $f(x)$ whose complement is countably compact. The c^*-continuity of f now gives an open $U\subset X$ containing x such that $f(U)\subset Y-\mathrm{cl}(V\cap W)$. Lemma 1 of [6] then shows $G(f)$ is closed.

Theorem 3.2 will now allow the regularity condition in Theorem 9 of [7] to be dropped.

Our next theorem parallels Theorem 2.1.

THEOREM 3.3. *For each $\alpha\in A$, let Y_α be a locally countably compact space and let $f_\alpha:X\to Y_\alpha$ be c^*-continuous from a first countable space X into Y_α. Then the function $f:X\to\prod_\alpha Y_\alpha$ defined by $f(x)=\{f_\alpha(x)\}$ is c^*-continuous.*

PROOF. Theorem 3.2 gives the graph of each f closed and the proof of Theorem 2.1 shows the graph of f is closed. Our conclusion follows from Theorem 4 of [7].

Since the concepts of compactness and countable compactness are the same in paracompact spaces, Example 2.2 shows that if $f:X\to Y_1\times Y_2$ is c^*-continuous, then each $f_i:X\to Y_i$ need not be c^*-continuous.

If $f:X\to Y$ is c^*-continuous, then $f:X\to f(X)$ need not be c^*-continuous as Example 2.12 shows. The next three theorems give conditions under which such a restriction on the range may be made.

THEOREM 3.4. *Let $f:X\to Y$ be a c^*-continuous function from a space X into a first countable Hausdorff space Y. Then $f:X\to f(X)$ is c^*-continuous.*

PROOF. The proof is similar to that of Theorem 2.13.

Note that the space Y of Example 2.12 is first countable, thereby showing the necessity of the Hausdorff condition in Theorem 3.4.

THEOREM 3.5. *Let $f:X\to Y$ be c^*-continuous and suppose $f(X)\subset Y$ is closed. Then $f:X\to f(X)$ is c^*-continuous.*

PROOF. The proof is analogous to that of Theorem 2.14.

THEOREM 3.6. *Let $f:X\to Y$ be a c^*-continuous function from the first countable space X into a space Y and suppose the graph of f is closed. Then $f:X\to f(X)$ is c^*-continuous.*

PROOF. As in the proof to Theorem 2.15, we can show the graph of $f:X\to f(X)$ is closed in $X\times f(X)$. By Theorem 4 of [7], $f:X\to f(X)$ is c^*-continuous.

For a function $f:X\to Y$, the graph function $g:X\to X\times Y$ is defined as $g(x)=(x,f(x))$ for each $x\in X$. In [5] conditions were given as to when c-continuous functions would have c-continuous graph functions. Conditions were also given as to when the c-continuity of the graph function would imply the c-continuity of the original function. Theorems 10 and 11 in [5] would, of course, apply to c^*-continuous functions since the concepts of compactness and countable compactness are equivalent in metric spaces. As in [5], we leave open the question of the existence of a c^*-continuous $f:X\to Y$ such that $g:X\to X\times Y$ is not c^*-continuous. Our final theorem gives conditions as to when the c^*-continuity of $g:X\to X\times Y$ implies the c^*-continuity of $f:X\to Y$.

THEOREM 3.7. *Let $f:X\to Y$ be a function, X countably compact and first countable and Y first countable. If the graph function $g:X\to X\times Y$ is c^*-continuous, then f is c^*-continuous.*

PROOF. Let $x\in X$ and let V be an open set containing $f(x)$ having countably compact complement. Then $P_Y^{-1}(V)$ is open in $X\times Y$ and, since X and $Y-V$ are countably compact and first countable, $X\times(Y-V)=P_Y^{-1}(V)$ is countably compact. Thus, $P_Y^{-1}(V)$ is an open set in $X\times Y$ having countably compact complement. Therefore, there exists an open U containing x such that $g(U)\subset P_Y^{-1}(V)$. It follows that $P_Y(g(U))=f(U)\subset V$ so that f is c^*-continuous.

The University of Arkansas,
Fayetteville, Arkansas, 72701

REFERENCES

[1] J. Dugundji, *Topology*, Allyn and Bacon, Boston, 1956.

[2] R.V. Fuller, *Relations Among Continuous and Various Noncontinuous Functions*, Pacific J. of Math., Vol. 25, No. 3 (1968) pp. 495—509.

[3] Karl R. Gentry and Hughes B. Hoyle, III, *c-continuous Functions*, The Yokohama Math. Journal, Vol. XVIII No. 2 (1970) pp. 71—76.

[4] Karl R. Gentry and Hughes B. Hoyle, III, *c-continuous Fundamental Groups*, Fundamenta Mathematicae 76 (1972) pp. 9—17.

[5] Paul E. Long and Michael D. Hendrix, *Properties of c-continuous Functions*, Yokohama Math. Journal, to appear.

[6] Paul E. Long, *Functions with Closed Graphs*, Amer. Math. Monthly, Vol. 76, No. 8, (1969) pp. 930—932.

[7] Young Soo Park, *c*-Continuous Functions*, J. Korean Math. Soc. 8 (1971) pp. 69—72.

[8] William J. Pervin and Norman Levine, *Connected Mappings of Hausdorff Spaces*, Proceedings of the Amer. Math. Soc., Vol. 9 (1958) pp. 488—495.