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A NOTE ON SEPARATION AXIOMS IN HYPERSPACES

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1. Introduction

For a topological space $(X, \mathscr{T})$, let 2^X be the space of all non-empty closed subsets of X with the finite topology (3, Definition 1.7) and $C(X)$, the subspace of 2^X consisting of all non-empty closed compact subsets of X with the finite topology.

In [3] E. Michael investigated many separation properties that are carried over from a topological space $(X, \mathscr{T})$ to $(2^X, 2^{\mathscr{T}})$ or to $(C(X), 2^{\mathscr{T}})$ where $2^{\mathscr{T}}$ stands for the finite topology. The present paper deals with some more separation properties namely almost regularity, semi regularity, almost normality and semi normality. As in [3], it is seen that all these separation axioms are carried over to 2^X is almost normal if and only if 2^X is almost regular and it is semi normal if and only if 2^X is semi regular. It is also shown that a space X is almost regular if and only if $C(X)$ is almost regular.

2. Notations and definitions

NOTATION 1. For a given topological space $(X, \mathscr{T})$ define

$$\mathfrak{a}(X)=\{A\subset X:\ A\neq\phi\}\ ,$$
$$2^X=\{A\subset X:\ A\neq\phi \text{ and } A \text{ is closed}\}\ ,$$
$$C(X)=\{A\in 2^X:\ A \text{ is compact}\}\ .$$

NOTATION 2. If $A_0, A_1, A_2, \cdots, A_n$ is any given system of subsets of X $(n\geq 0)$, then define

$$B(A_0, A_1, \cdots, A_n)=\{F\in 2^X: F\subset A_0, F\cap A_i\neq\phi \text{ for } i\leq n\}$$

Without loss of generality it can be assumed that $A_i\subset A_0$ for $i=1,2,\cdots,n$, since

$$B(A_0, A_1, \cdots, A_n)=B(A_0, A_1\cap A_0, \cdots, A_n\cap A_0)$$

NOTATION 3. When there is no confusion $\mathrm{Cl}(F)$ denotes the closure of F and $\mathrm{Int}(F)$ denotes the interior of F. Otherwise we shall write $\mathscr{T}$-cl F to indicate that the closure of F is taken with respect to $\mathscr{T}$. Similar meaning for $\mathscr{T}$-int F.

DEFINITION 1. (2, page 160) For a given topological space $(X, \mathscr{T})$ the collection of all sets of the form $B(G_0, G_1, \cdots, G_n)$ with $G_0, G_1, \cdots, G_n$ in $\mathscr{T}$, form a basis for a topology $2^{\mathscr{T}}$ on 2^X, called *the finite topology* [3] or *exponential topology* [2].

REMARK 1. The collection of all sets of the form $B(G_0, G_1, \cdots, G_n)$ with $G_0, G_1, \cdots G_n$ in $\mathscr{T}$ where $B(G_0, \cdots, G_n)=\{F\in \mathfrak{A}(X): F\subset G_0,$ and $F\cap G_i\neq\phi$ for $i=1,2\cdots n$ form a basis for a topology on $\mathfrak{A}(X)$, denoted again by $2^{\mathscr{T}}$. It is easily seen that the finite topology on 2^X is the relative topology induced on 2^X by the finite topology on $\mathfrak{A}(X)$ (3, 5.2.2).

Now we mention two results from [2] and [3] which we use very frequently.

LEMMA 1. (a) *For arbitrary subsets* $A_0, A_1, \cdots, A_n$ *and* $B_0, B_1, \cdots, B_m$ *of* X

$B(A_0, A_1, \cdots, A_n)\subset B(B_0, B_1, \cdots, B_m)$ *if and only if* $\bigcup_{i=0}^{n} A_i\subset\bigcup_{j=0}^{m} B_j$ *and for each* B_j *there exist an* A_i *such that* $A_i\subset B_j$ (see 3, 2.3.1)

(b) *If* $A_i\subset A_0$ *for* $i=1,2,\cdots,n$

then

$$2^{\mathscr{T}}\text{-cl } B(A_0, A_1, \cdots, A_n)=B(\mathscr{T}\text{-cl } A_0, \mathscr{T}\text{-cl } A_1, \cdots, \mathscr{T}\text{-cl } A_n)$$

and $2^{\mathscr{T}}\text{-int } B(A_0, A_1, \cdots, A_n)=B(\mathscr{T}\text{-int } A_0, \cdots, \mathscr{T}\text{-int } A_n)$

(see 2, pages 160—162).

We next collect the definitions of the separation properties that we have used in this paper.

DEFINITION 2. A set is said to be *regularly closed* [2] if it is the closure of its own interior or equivalently if it is the closure of some open set.

DEFINITION 3. A space $(X, \mathscr{T})$ is said to be *almost regular* [4] if for every regularly closed set A and each point x not belonging to A there exist disjoint open sets U and V containing A and x respectively. Equivalently a space X is *almost regular* if for each x in X and every open set U containing x there is an open set V such that

$$x\in V\subset \text{Cl } V\subset \text{Int cl } U .$$

DEFINITION 4. A space $(X, \mathscr{T})$ is said to be *semi regular* [4] if for each point x in X and any open set U containing x, there exists an open set V such that

$$x\in V\subset \text{Int cl } V\subset U .$$

DEFINITION 5. A space $(X, \mathscr{T})$ is said to be *almost normal* [5] if for every pair of disjoint sets A and B, one of which is closed and the other is regularly closed, there exist disjoint open sets U and V such that $A \subset U$ and $B \subset V$. Equivalently $(X, \mathscr{T})$ is *almost normal* if for every closed set A and every open set $U \supset A$, there exist an open set V such that

$$A \subset V \subset \mathrm{Cl}\ V \subset \mathrm{Int\ cl}\ U\ .$$

DEFINITION 6. A space $(X, \mathscr{T})$ is said to be *semi normal* [5] if for every closed set A and every open set U containing A, there exists an open set V such that

$$A \subset V \subset \mathrm{Int\ cl}\ V \subset U.$$

DEFINITION 7. A space $(X, \mathscr{T})$ is said to be an *E_1-space* [1] if every point is a countable intersection of closed neighbourhoods of that point.

REMARK 2. It is obvious that a space is E_1 if at each point x_0, there exist a countable number of basic open neighbourhoods say $\{G_i\}_{i=1}^{\infty}$ of that point such that

$$\{x_0\} = \cap_{i=1}^{\infty} \mathrm{Cl}\ G_i\ .$$

3. We first prove the equivalence of almost normality of a topological space $(X, \mathscr{T})$ with almost regularity of the hyperspace $(2^X, 2^{\mathscr{T}})$.

THEOREM 1. *For a T_1-space $(X, \mathscr{T})$, the following are equivalent*

(i) *$(X, \mathscr{T})$ is almost normal*

(ii) *$(2^X, 2^{\mathscr{T}})$ is almost regular.*

PROOF. Assume that X is almost normal. Let $F_0 \in 2^X$ and $\mathscr{U}$ a $2^{\mathscr{T}}$-open set containing F_0. We have to show that there exists a $2^{\mathscr{T}}$-open set $\mathscr{V}$ such that

$$F_0 \in \mathscr{V} \subset 2^{\mathscr{T}}\text{-cl}\ \mathscr{V} \subset 2^{\mathscr{T}}\text{-int}\ 2^{\mathscr{T}}\text{-cl}\ \mathscr{U}.$$

Without loss of generality we may assume that $\mathscr{U}$ is a basic open set in $(2^X, 2^{\mathscr{T}})$ i.e. we may assume that $\mathscr{U} = B(G_0, G_1, \cdots, G_n)$ where $G_0, G_1, \cdots, G_n$ are $\mathscr{T}$-open sets.

Since $F_0 \in \mathscr{U}$, we have $F_0 \subset G_0$ and $F_0 \cap G_i \neq \phi$ for $i = 1, 2, \cdots, n$. Assume that

$$q_i \in F_0 \cap G_i \text{ for } i = 1, 2, \cdots, n.$$

Since $(X, \mathscr{T})$ is almost normal, there exist $\mathscr{T}$-open sets $U_0, U_1, U_2, \cdots, U_n$ such that

$$F_0 \subset U_0 \subset \mathscr{T}\text{-cl}\ U_0 \subset \mathscr{T}\text{-int} \mathscr{T}\text{-cl}\ G_0$$

and

$$q_i \in U_i \subset \mathscr{T}\text{-cl } U_i \subset \mathscr{T}\text{-int } \mathscr{T}\text{-cl } U_i \text{ for } i=1,2,\cdots,n.$$

Define $\mathscr{V}=B(U_0, U_1, \cdots, U_n)$.

Then by Lemma 1(*b*)

$$\begin{aligned} 2^{\mathscr{T}}\text{-cl}\,\mathscr{V} &= 2^{\mathscr{T}}\text{-cl } B(U_0, U_1, \cdots U_n) \\ &= B(\mathscr{T}\text{-cl } U_0, \mathscr{T}\text{-cl } U_1, \cdots, \mathscr{T}\text{-cl } U_n) \\ &\subset B(\mathscr{T}\text{-int}\mathscr{T}\text{-cl } G_0, \mathscr{T}\text{-int}\mathscr{T}\text{-cl } G_1, \cdots, \mathscr{T}\text{-int}\mathscr{T}\text{-cl } G_n) \\ &\subset 2^{\mathscr{T}}\text{-int } 2^{\mathscr{T}}\text{-cl } B(G_0, G_1, \cdots, G_n) \\ &\subset 2^{\mathscr{T}}\text{-int } 2^{\mathscr{T}}\text{-cl}\,\mathscr{U} . \end{aligned}$$

Thus

$$F_0 \in \mathscr{V} \subset 2^{\mathscr{T}}\text{-cl } \mathscr{V} \subset 2^{\mathscr{T}}\text{-int } 2^{\mathscr{T}}\text{-cl}\,\mathscr{U}.$$

Conversely, let $(2^X, 2^{\mathscr{T}})$ be almost regular. Let F_0 be a $\mathscr{T}$-closed set and U, $\mathscr{T}$-open set containing F_0. It is enough to show that there exists a $\mathscr{T}$-open set such that

$$F_0 \subset V \subset \mathscr{T}\text{-cl } V \subset \mathscr{T}\text{-int } \mathscr{T}\text{-cl } U.$$

Now $B(U)$ is a $2^{\mathscr{T}}$-open set containing F_0. By almost regularity of $(2^X, 2^{\mathscr{T}})$, there exists a $2^{\mathscr{T}}$-basic open set $B(G_0, G_1, G_2, \cdots, G_n)$ such that
$F_0 \in B(G_0, G_1, \cdots, G_n) \subset 2^{\mathscr{T}}\text{-cl } B(G_0, G_1, \cdots, G_n) \subset 2^{\mathscr{T}}\text{-int } 2^{\mathscr{T}}\text{-cl}B(U)$ i.e. $F_0 \in B(G_0, G_1, \cdots, G_n) \subset B(\mathscr{T}\text{-cl } G_0, \mathscr{T}\text{-cl } G_1, \cdots, \mathscr{T}\text{-cl } G_n) \subset B(\mathscr{T}\text{-int } \mathscr{T}\text{-cl } U)$
Now $\mathscr{T}\text{-cl } G_0 \cap \mathscr{T}\text{-cl } G_i = \mathscr{T}\text{-cl } G_i \neq \phi$. Hence

$$\mathscr{T}\text{-cl } G_0 \in B(\mathscr{T}\text{-cl } G_0, \mathscr{T}\text{-cl } G_1, \cdots, \mathscr{T}\text{-cl } G_n) \subset B(\mathscr{T}\text{-int } \mathscr{T}\text{-cl } U)$$

i.e., $\mathscr{T}\text{-cl } G_0 \subset \mathscr{T}\text{-int } \mathscr{T}\text{-cl } U$.

Clearly $F_0 \subset G_0 \subset \mathscr{T}\text{-cl } G_0 \subset \mathscr{T}\text{-int}\mathscr{T}\text{-cl } U$.

Taking $V=G_0$, we have the desired result.

4. We next prove the equivalence of semi normality of a topological space $(X, \mathscr{T})$ with semi regularity of the hyperspace $(2^X, 2^{\mathscr{T}})$.

THEOREM 2. *For a T_1-space $(X, \mathscr{T})$ the following are equivalent*

(i) *$(X, \mathscr{T})$ is semi normal,*

(ii) *$(2^X, 2^{\mathscr{T}})$ is semi regular .*

PROOF. Assume that $(X, \mathscr{T})$ is semi normal. Let $F_0 \in 2^X$ and $\mathscr{U}$ an $2^{\mathscr{T}}$-open set containing F_0. We have to show that there exists a $2^{\mathscr{T}}$-open set $\mathscr{V}$ such that

$$F_0 \in \mathscr{V} \subset 2^{\mathscr{T}}\text{-int } 2^{\mathscr{T}}\text{-cl } \mathscr{V} \subset \mathscr{U}.$$

Without loss of generality, we may assume that $\mathscr{U}$ is a $2^{\mathscr{T}}$-basic open set i.e. $\mathscr{U}=B(G_0, G_1, G_2, \cdots, G_n)$ for some $\mathscr{T}$-open sets

$$G_0, G_1, G_2, \cdots, G_n.$$

Since $F_0 \in \mathscr{U}$, we have $F_0 \subset G$ and $F_0 \cap G_i \neq \phi$ for $i=1, 2, \cdots, n$. Assume that

$$q_i \in F_0 \cap G_i \text{ for } i=1, 2, \cdots, n.$$

By semi normality of $(X, \mathscr{T})$, there exist $\mathscr{T}$-open sets $U_0, U_1, \cdots, U_n$ such that

$$F_0 \subset U_0 \subset \mathscr{T}\text{-int}\mathscr{T}\text{-cl } U_0 \subset G_0$$

and

$$q_i \in U_i \subset \mathscr{T}\text{-int } \mathscr{T}\text{-cl } U_i \subset G_i \text{ for } i=1, 2, \cdots, n.$$

Define

$$\mathscr{V}=B(U_0, U_1, \cdots, U_n) .$$

Then by Lemma 1(*b*)

$$\begin{aligned} 2^{\mathscr{T}}\text{-int } 2^{\mathscr{T}}\text{-cl } \mathscr{V} &= 2^{\mathscr{T}}\text{-int } 2^{\mathscr{T}}\text{-cl } B(U_0, U_1, \cdots, U_n) \\ &= B(\mathscr{T}\text{-int } \mathscr{T}\text{-cl } U_0, \cdots, \mathscr{T}\text{-int } \mathscr{T}\text{-cl } U_n) \\ &\subset B(G_0, G_1, \cdots, G_n) \subset \mathscr{U}. \end{aligned}$$

Hence

$$F_0 \in \mathscr{V} \subset 2^{\mathscr{T}}\text{-int } 2^{\mathscr{T}}\text{-cl } \mathscr{V} \subset \mathscr{U}.$$

Conversely let $(2^X, 2^{\mathscr{T}})$ be semi regular. Let F_0 be a closed set and U an $\mathscr{T}$-open set containing F_0. Then $F_0 \in B(U)$. By semi regularity of $(2^X, 2^{\mathscr{T}})$ there exists a $2^{\mathscr{T}}$-basic open set $B(G_0, G_1, \cdots, G_n)$ such that

$$F_0 \in B(G_0, G_1, \cdots, G_n) \subset 2^{\mathscr{T}}\text{-int } 2^{\mathscr{T}}\text{-cl } B(G_0, G_1, \cdots, G_n) \subset B(U)$$

or $\quad F_0 \in B(G_0, G_1, \cdots, G_n) \subset B(\mathscr{T}\text{-int } \mathscr{T}\text{-cl } G_0, \cdots, \mathscr{T}\text{-int } \mathscr{T}\text{-cl } G_n) \subset B(U)$.

Then, obviously $F_0 \subset G_0 \subset \mathscr{T}\text{-int } \mathscr{T}\text{-cl } G_0 \subset U$ the last relationship following from Lemma 1(*a*) .

We can now deduce the well known theorem (2, pages 170—171) namely a T_1-space $(X, \mathscr{T})$ is normal if and only if $(2^X, 2^{\mathscr{T}})$ is regular.

COROLLARY 1. *For a T_1-space $(X, \mathscr{T})$ the following are equivalent*

(i) $(X, \mathscr{T})$ *is normal*

(ii) $(2^X, 2^{\mathscr{T}})$ *is regular.*

PROOF. It is known that a space is regular if and only if it is semi regular and almost regular [4] and it is normal if and only if it is semi normal and almost normal [5]. The proof is obvious in view of Theorems 1 and 2.

5. We next consider the hyperspace $(C(X), 2^{\mathscr{T}})$ and see how far the separation properties almost regularity and semi regularity are carried over from the space $(X, \mathscr{T})$ to $(C(X), 2^{\mathscr{T}})$ and vice versa.

LEMMA 2. *If $(X, \mathscr{T})$ is almost regular, then given a compact set A and an open set U containing it, there exists an open set V such that $A \subset U \subset \mathrm{Cl}\ U \subset \mathrm{Int}$ $\mathrm{cl}\ V$.*

PROOF. The proof is similar to the corresponding result for compact sets in regular spaces.

THEOREM 3. *A topological space $(X, \mathscr{T})$ is almost regular if and only if $(C(X), 2^{\mathscr{T}})$ is almost regular.*

PROOF. Let $(X, \mathscr{T})$ be almost regular. Let $F_0 \in C(X)$ and $\mathscr{U} = B(G_0, G_1, \cdots, G_n)$ be a $2^{\mathscr{T}}$-basic open set containing F_0. Then $F_0 \subset G_0$ and $F_0 \cap G_i \neq \phi$ for $i=1, 2, \cdots, n$. Assume $q_i \in F_0 \cap G_i$ for $i=1, 2, \cdots, n$. By Lemma 2, there exists an $\mathscr{T}$-open set U_0 such that

$$F_0 \subset U \subset \mathscr{T}\text{-cl } U_0 \subset \mathscr{T}\text{-int } \mathscr{T}\text{-cl } G_0 .$$

By almost regularity of X, there exist open sets $U_1, U_2, \cdots, U_n$ such that

$$q_i \in U_i \subset \mathscr{T}\text{-cl } U_i \subset \mathscr{T}\text{-int } \mathscr{T}\text{-cl } G_i \text{ for } i=1, 2, \cdots, n.$$

Then by a similar argument as in Theorem 1, We have

$$F_0 \in B(U_0, U_1, \cdots, U_n) \subset 2^{\mathscr{T}}\text{-cl } B(U_0, U_1, \cdots, U_n) \subset 2^{\mathscr{T}}\text{-int } 2^{\mathscr{T}}\text{-cl}\, \mathscr{U} .$$

Conversely assume$(C(X), 2^{\mathscr{T}})$ is almost regular. Let $x_0 \in U$, U a $\mathscr{T}$-open set. Then $\{x_0\}$ is compact and $\{x_0\} \in B(U)$. Hence there exists a $2^{\mathscr{T}}$-basic open set $B(G_0, G_1, \cdots, G_n)$ such that

$$\{x_0\} \in B(G_0, G_1, \cdots, G_n) \subset 2^{\mathscr{T}}\text{-cl } B(G_0, \cdots, G_n) \subset 2^{\mathscr{T}}\text{-int } 2^{\mathscr{T}}\text{-cl } B(U).$$

Then obviously (See Theorem 1)

$$x_0 \in G_0 \subset \mathscr{T}\text{-cl } G_0 \subset \mathscr{T}\text{-int } \mathscr{T}\text{-cl } U.$$

THEOREM 4. *If $(C(X), 2^{\mathscr{T}})$ is semi regular then $(X, \mathscr{T})$ is semi regular.*

PROOF. Let $x_0 \in U$, U a $\mathscr{T}$-open set. Then $\{x_0\} \in C(X)$ and $\{x_0\} \in B(U)$. By semi regularity of $C(X)$ there exists a basic open set $B(G_0, G_1, \cdots, G_n)$ such that

$$\{x_0\} \in B(G_0, G_1, \cdots, G_n) \subset 2^{\mathscr{T}}\text{-int } 2^{\mathscr{T}}\text{-cl } B(G_0, G_1, \cdots, G_n) \subset B(U).$$

Then obviously

$$x_0 \in G_0 \subset \mathscr{T}\text{-int } \mathscr{T}\text{-cl } G_0 \subset U.$$

6. Lastly we consider the property of a space being an E_1-space.

THEOREM 5. *For a T_1-space $(X, \mathscr{T})$, $(X, \mathscr{T})$ is an E_1-space if $(2^X, 2^{\mathscr{T}})$ is an E_1-space.*

PROOF. Let $x_0 \in X$. Since $(X, \mathscr{T})$ is T_1, $\{x_0\} \in 2^X$. By hypothesis there exist a countable number of $2^{\mathscr{T}}$-basic open neighbourhoods $B(G_0^i, G_1^i, \cdots, G_{n_i}^i)$ i varying from 1 to ∞ of $\{x_0\}$ such that

$$\{x_0\} = \cap_{i=1}^{\infty} 2^{\mathscr{T}}\text{-cl } B(G_0^i, G_1^i, \cdots, G_{n_i}^i) .$$

Define for each i from 1 to ∞

$$G_i = G_0^i \cap G_1^i \cap \cdots \cap G_{n_i}^i .$$

Then each G_i is a $\mathscr{T}$-open neighbourhood of x_0. We claim that

$$\{x_0\} = \cap_{i=1}^{\infty} \mathscr{T}\text{-cl } G_i .$$

For if $t \neq x_0$ belongs to $\cap_{i=1}^{\infty} \mathscr{T}\text{-cl } G_i$, then obviously $\{t\} \in 2^{\mathscr{T}}\text{-cl } B(G_0^i, G_1^i, \cdots, G_{n_i}^i)$ for each i from 1 to ∞ and hence

$$\{t\} \in \cap_{i=1}^{\infty} 2^{\mathscr{T}}\text{-cl } B(G_0^i, G_1^i, \cdots, G_n^i)$$

a contradiction.

COROLLARY 2. *A topological space $(X, \mathscr{T})$ is an E_1-space if the hyperspace $(C(X), 2^{\mathscr{T}})$ is an E_1-space.*

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