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NECESSARY AND SUFFICIENT CONDITIONS FOR A LINEAR FUNCTIONAL TO BE CONTINUOUS

By T.R. Hamlett

The following classical theorem which can be found in [6, Theorem 1.18, page 14] gives necessary and sufficient conditions for a linear functional to be continuous.

THEOREM 1. *Let Λ be a linear functional on a topological vector space X. Assume $\Lambda x \neq 0$ for some x in X. Then the following are equivalent:*

(a) *Λ is continuous.*

(b) *The null space $N(\Lambda)$ is closed.*

(c) *$N(\Lambda)$ is not dense in X.*

(d) *Λ is bounded on some neighborhood V of O.*

The purpose of this paper is to give a more general version of the above theorem using the notion of a *semi-open* set. Norman Levine [5] defined a set A in a topological space X to be semi-open if there exists an open set U such that $U \subset A \subset \mathrm{Cl}(U)$, where $\mathrm{Cl}(U)$ denotes the closure of U. Henceforth we will use *nbd* to abbreviate *neighborhood*.

In order to achieve our purpose we will need some preliminary definitions and propositions. The following definition is slightly different from Definition 1 of [1].

DEFINITION 1. A *semi-nbd* of a point p in a topological space X is a semi-open set which contains p.

Levine [5] defined a function f from a topological space X into a topological space Y to be *semi-continuous* provided $f^{-1}(V)$ is semi-open in X for every open set V in Y. It will be convenient to have the following pointwise definition.

DEFINITION 2. A function f from a topological space X into a topological space Y is said to be *semi-continuous at p in X* if for every nbd V of $f(p)$ there exists a semi-nbd A of p such that $f(A) \subset V$.

PROPOSITION 1. *A function $f: X \to Y$ is semi-continuous if and only if it is semi-continuous at every point of X.*

PROOF. Theorem 12 of [5]

A function $f: X \to Y$ is said to be *pre-semi-open* [4, Definition 1.2] if $f(A)$ is semi-open in Y for every semi-open set A in X.

Now let X denote a topological vector space. Associate to each a in X and to each scalar $\lambda \neq 0$ the translation operator $T(a)$ and the multiplication operator $M(\lambda)$, defined by

$$T(a)x = a + x, \quad M(\lambda)x = \lambda x \quad (x \in X).$$

PROPOSITION 2. *$T(a)$ and $M(\lambda)$ are pre-semi-open.*

PROOF. Since $T(a)$ and $M(\lambda)$ are homeomorphisms, they are continuous and open. Hence the result follows from [4, Theorem 1.8].

PROPOSITION 3. *Let X and Y be topological vector spaces. If $\Lambda: X \to Y$ is linear and semi-continuous at O, then Λ is semi-continuous.*

PROOF. Let $x \in X$ and let W be a nbd of Λx in Y. Then $W - \Lambda x$ is a nbd of O in Y and therefore there exists a semi-nbd A of O in X such that $\Lambda(A) \subset W - \Lambda x$. This implies $\Lambda(A+x) \subset W$. By Proposition 2, $A+x$ is a semi-nbd of x and the proof is complete.

A subset of a topological space is said to be *semi-closed* [3] if its complement is semi-open. We are now ready to prove a more general version of Theorem 1.

THEOREM 2. *Let Λ be a linear functional on a topological vector space X. Assume $\Lambda x \neq 0$ for some x in X. Then the following are equivalent:*

(a) *Λ is continuous.*

(b) *Λ is semi-continuous.*

(c) *The null space $N(\Lambda)$ is semi-closed.*

(d) *$N(\Lambda)$ is not dense in X.*

(e) *Λ is bounded on some semi-nbd of O.*

PROOF. (a) implies (b) is clear.

It is shown in [2, Theorem 1.3] that a function $f: Y \to Z$, where Y and Z are topological spaces, is semi-continuous if and only if the inverse image of every closed set in Z is semi-closed in Y. It now follows that $N(\Lambda) = \Lambda^{-1}\{0\}$ is semi-closed if Λ is semi-continuous, and we have (b) implies (c).

If we assume $N(\Lambda)$ is semi-closed, then the complement of $N(\Lambda)$ is a non-empty semi-open set and hence must have a non-empty interior. Thus $N(\Lambda)$ is not dense in X and (c) implies (d).

Theorem 1 contains the fact that (d) implies (a).

We now show that (e) is equivalent to (b). If we assume Λ is semi-continuous, then Λ must be continuous by the previous part of the proof. Hence Λ is bounded on a nbd of O by Theorem 1 and we have (b) implies (e). On the other hand, if Λ is bounded on some semi-nbd A of O, we have

$$|\Lambda x| < M (x \in A, \ M < \infty).$$

If $x > 0$ and $W = (r/M)A$, then $|\Lambda x| < r$ for every $x \in W$. Note that W is a semi-nbd of O by Proposition 2. Hence Λ is semicontinuous by Proposition 3, and (e) implies (b).

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University of Arkansas,
Fayetteville, Arkansas 72701

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