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A CERTAIN SEMIGROUP ASSOCIATED WITH A TOPOLOGICAL SPACE

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1. Introduction

For a topological space there correspond, in a very natural way, many kinds of semigroups; for instance, those of all continuous self-mappings, of all closed self-mappings, of all connected self-mappings, of all closed relations, of all closed relations with compact ranges, etc. These semigroups determine the original topological space in the sense that two topological spaces in a certain class of topological spaces are homeomorphic if and only if their corresponding semigroups are isomorphic. (cf. [1] theorem 3.1, [2] theorem 2.10 and theorem 3.5, [3] theorem 3.3, [4] theorem 2.3). In all of those semigroups the constant mappings or 'subconstant mappings', i.e., constant mappings defined on subsets of the whole space, play a great role. In the present paper we consider a semigroup of certain subconstant mappings of a topological space and obtain the above result for the whole class of topological spaces.

Let $(X, \mathscr{T})$ be a topological space. Let $\mathscr{T}(X)$ be the family of all subconstant mappings with open domains and an empty mapping ϕ, explicitly, $\mathscr{T}(X)=\{U\times\{x\}\,|\,U\in\mathscr{T},\ x\in X\}$. Since $\mathscr{T}(X)$ is closed under composition, $\circ$, as relations, it is a semigroup with composition, $\circ$, as a binary operation and the empty mapping is the zero element of it. We will write U_x instead of $U\times\{x\}$ and 0 instead of ϕ.

In section 2 we investigate the structure of $\mathscr{T}(X)$, and in section 3 we give our main theorems: i) there is a covariant functor from the category of the topological spaces with open injections as morphisms to the category of the semigroups with injective homomorphisms as morphisms; ii) there is a covariant functor from the category of the semigroups of the form $\mathscr{T}(X)$ with surjective homomorphisms as morphisms to the category of the topological spaces with open surjections as morphisms; iii) as a consequence of i) and ii), two topological spaces are homeomorphic if and only if their corresponding semigroups are isomorphic.

All terms concerning semigroups are found in [5], those of topology in [6], and those of category in [7].

2. The structure of $\mathscr{T}(X)$

First we need the following

LEMMA 2.1. (a) $U_x \circ V_y = \begin{cases} V_x & \text{if } y \in U \\ 0 & \text{if } y \notin U. \end{cases}$

(b) $U_x \circ U_x = U_x$ *iff* $x \in U$.

(c) $U_x \circ U_x = 0$ *iff* $x \notin U$.

PROOF. Obvious.

For $x \in X$, let $[x] = \{U_x | U \in \mathscr{T}\}$. Then we have the following theorem concerning right ideals of $\mathscr{T}(X)$.

THEOREM 2.2. (a) *Every* $[x]$ *is a* 0*-minimal right ideal of* $\mathscr{T}(X)$, *and every* 0*-minimal right ideal is of this type.*

(b) $[x]\,[y] = [x]$, *and hence the family of all* 0*-minimal right ideals form a left zero semigroup with the set product as a binary operation.*

PROOF. (a) Since $U_x \circ V_y = V_x$ $(y \in U)$, or 0 $(y \notin U)$, $[x]\mathscr{T}(X) \subset [x]$, that is, $[x]$ is a right ideal of $\mathscr{T}(X)$. If $\mathscr{A}$ is a right ideal $(\neq 0)$ contained in $[x]$, then there is a nonempty $U \in \mathscr{T}$ such that $U_x \in \mathscr{A}$. Taking $y \in U$, $V_x = U_x \circ V_y \subset \mathscr{A}$ V_y $\subset \mathscr{A}$ for every $V \in \mathscr{T}$. Hence $[x] \subset \mathscr{A}$. Therefore $[x]$ is a 0-minimal right ideal of $\mathscr{T}(X)$.

Next, if $\mathscr{B}$ is a 0-minimal right ideal of $\mathscr{T}(X)$, then there is nonzero $U_x \in \mathscr{B}$. Since $[x] \cap \mathscr{B}$ is a right ideal of $\mathscr{T}(X)$ contained in $[x]$ and contains nonzero U_x, $[x] \subset \mathscr{B}$ by the minimality of $[x]$, and $\mathscr{B} \subset [x]$ by the minimality of $\mathscr{B}$. Hence $\mathscr{B} = [x]$.

(b) $[x]\,[y] \subset [x] = X_x[y] \subset [x]\,[y]$ because $[x] = X_x[y]$.

With the obvious modifications of the definitions in [5, page 25]. the above theorem can be rephrased as:

$\mathscr{T}(X)$ *is a left zero semigroup of all* 0*-minimal right ideals.*

in the sense that $\mathscr{T}(X)$ is a 0-disjoint union of all 0-minimal right ideals of it which form a left zero semigroup with the set product as a binary operation.

Next we have an analogous theorem with left ideals. For $U(\neq \phi) \in \mathscr{T}$, let $[U] = \{U_x | x \in X\} \cup \{0\}$. Then we have the following

THEOREM 2.3. (a) *Every* $[U]\,(U\neq\phi)$ *is a* 0-*minimal left ideal of* $\mathscr{T}(X)$ *and every* 0-*minimal left ideal of* $\mathscr{T}(X)$ *is of this type.*

(b) $[U]\,[V]=[V]$, *and hence the family of all* 0-*minimal left ideals of* $\mathscr{T}(X)$ *forms a right zero semigroup with the set product as a binary operation.*

PROOF. (a) Similar as in theorem 2.2.

(b) $[U]\,[V]\subset[V]=[U]\,V_u\subset[U]\,[V]$ if we take $u\in U$.

As in theorem 2.2, this theorem can be rephrased as:

$\mathscr{T}(X)$ *is a right zero semigroup of all* 0-*minimal left ideals of it.*

In the next section we use the following property of $\mathscr{T}(X)$.

THEOREM 2.4. $\mathscr{T}(X)$ *is* 0-*simple.*

PROOF. Clearly $\mathscr{T}(X)\mathscr{T}(X)\supset[X]\,[x]=\mathscr{T}(X)\neq 0$. Next if $\mathscr{A}$ is a nonzero ideal, then $\mathscr{A}$ contains a nonzero U_x. Taking $y\in U$,

$[x]=U_x[y]\subset\mathscr{A}[y]\subset\mathscr{A}$; so $\mathscr{T}(X)=[X]\,[x]\subset[X]\mathscr{A}\subset\mathscr{A}$.

Hence $\mathscr{A}=\mathscr{T}(X)$. Therefore $\mathscr{T}(X)$ is 0-simple.

3. Functors

To each open mapping $h:(X,\mathscr{T})\longrightarrow(Y,\mathscr{U})$ is assigned a mapping $h_*:\mathscr{T}(X)\longrightarrow\mathscr{U}(Y)$ which is defined, in a very natural way, by $h_*(U_x)=h(U)_{h(x)}$ for $U_x\in\mathscr{T}(X)$. This assignment gives rise to a covariant functor from the category of the topological spaces with open injections as morphisms to the category of the semigroups with injective homomorphisms as morphisms, as is shown below.

THEOREM 3.1. (a) *If* h *is an open injection,* h_* *is an injective homomorphism.*

(b) *If* $k:(Y,\mathscr{T})\longrightarrow(Z,\mathscr{U})$ *is another open injection, then* $(k\circ h)_*=k_*\circ h_*$.

(c) *If* $i:(X,\mathscr{T})\longrightarrow(X,\mathscr{T})$ *is the identity, then* i_* *is also the identity.*

PROOF. (a) Suppose that $h_*(U_x)=h_*(V_y)$ i.e., $h(U)_{h(x)}=h(V)_{h(y)}$. If either $h(U)=\phi$ or $h(V)=\phi$, then both of them are empty; so $U=\phi=V$, and hence $U_x=0=V_y$. If neither $h(U)$ nor $h(V)$ is empty, then $h(U)=h(V)$ and $h(x)=h(y)$. By the injectiveness of h, $U=V$ and $x=y$, and so $U_x=V_y$. Hence h_* is injective.

Next, since h is injective, $h(y)\in h(U)$ iff $y\in U$.

$$h_*(U_x)\circ h_*(V_y)=h(V)_{h(x)}\circ h(V)_{h(y)}=\begin{cases} h(V)_{h(x)} & \text{if } h(y)\in h(U)\\ 0 & \text{if } h(y)\notin h(U)\end{cases}$$

$$h_*(U_x\circ V_y)=\begin{cases} h_*(V_x)=h(V)_{h(x)} & \text{if } y\in U\\ h_*(0)=0 & \text{if } y\notin U.\end{cases}$$

Hence $h_*(U_x \circ V_y)=h_*(U_x)\circ h_*(V_y)$, and so h_* is a homomorphism.

(b) $(k\circ h)_*(U_x)=(k\circ h(U))_{k\circ h(x)}=k_*(h(U)_{h(x)})=k_*\circ h_*(U_x)$.

(c) $i_*(U_x)=i(U)_{i(x)}=U_x$.

LEMMA 3.2. *If* H: $\mathscr{T}(X)\longrightarrow\mathscr{U}(Y)$ *is a homomorphism and* $H(U_x)=V_y(U\neq\phi\neq V)$, *then* (a) $H^{-1}(0)=0$ *and* (b) $x\in U$ *iff* $y\in V$.

PROOF. (a) $H^{-1}(0)$ is an ideal of $\mathscr{T}(X)$ and does not contain U_x. Since $\mathscr{T}(X)$ is 0-simple (theorem 2.4), $H^{-1}(0)=0$.

(b) $x\in U$ iff $U_x\circ U_x=U_x$ iff $V_y\circ V_y=V_y$ iff $y\in V$.

LEMMA 3.3. *If* H: $\mathscr{T}(X)\longrightarrow\mathscr{T}(Y)$ *is a surjective homomorphism and* $H(U_x)=V_y(U\neq\phi)$, *then* (a) $H([x])=[y]$ *and* (b) $H([U])=[V]$.

PROOF. By lemma 3.2(a), V is not empty. Since a surjective homomorphism preserves right (left) ideals, $H([x])$ $(H([U]))$ is a right (left) ideal which contains V_y. Since $[y]$ ($[V]$ resp.) is 0-minimal right (left resp.) ideal of $\mathscr{U}(Y)$ containing V_y, $[y]\subset H([x])$ ($[V]\subset H([U])$ resp.). Next since $H^{-1}([y])$ $(H^{-1}([V])$ resp.) is a right (left, resp.) ideal of $\mathscr{T}(X)$ containing U_x, $[x]\subset H^{-1}([y])$ ($[U]\subset H^{-1}([V])$ resp.); so $H([x])\subset[y]$. ($H([U])\subset[V]$ resp.) Hence $H([x])=[y]$. ($H([U])=[V]$ resp.)

By the virtue of lemma 3.3, to every surjective homomorphism $H:\mathscr{T}(X)\longrightarrow\mathscr{U}(Y)$ there corresponds a mapping $H_*:(X,\mathscr{T})\longrightarrow(Y,\mathscr{U})$ defined by $H_*(x)=y$ iff $H(U_x)=V_y$, or equivalently iff $H([x])=[y]$. The next theorem shows that this correspondence establishes a covariant functor from the category of the semigroups of the form $\mathscr{T}(X)$ with surjective homomorphisms as morphisms to the categery of the topological spaces with open surjections as morphisms.

THEOREM 3.4. (a) *If* H *is a surjective homomorphism, then* H_* *is an open surjection such that* $H(U_x)=H_*(U)_{H_*(x)}$.

(b) *If* K *is another surjective homomorphism from* $\mathscr{U}(Y)$ *onto* $\mathscr{V}(Z)$, *then* $(K\circ H)_*=K_*\circ H_*$.

(c) *If* $I:\mathscr{T}(X)\longrightarrow\mathscr{T}(X)$ *is the identity, then* I_* *is also the identity.*

PROOF. (a) For any $y\in Y$, $Y_y\in\mathscr{U}(Y)$; since H is surjective, $Y_y=H(U_x)$ for some $U_x\in\mathscr{T}(X)$. Thus $y=H_*(x)$ and hence H_* is surjective.

Next, let $U\in\mathscr{T}$ and $U\neq\phi$ and let $V_y=H(U_x)$. For any $u\in U$, $U_u\circ U_u=U_u$. Since $H([U])=[V]$, by lemma 3.3, $H(U_u)=V_{H_*(u)}$, and hence $V_{H_*(u)}\circ V_{H_*(u)}=V_{H_*(u)}$, which implies $H_*(u)\in V$. Therefore $H_*(U)\subset V$.

Conversely, if $y\in V$ there is an $x\in X$ such that $H(U_x)=V_y$. By lemma 3.2 (b), $x\in U$ and hence $y=H_*(x)\in H_*(U)$. Thus $V\subset H_*(U)$ Therefore $H_*(U)=V\in\mathscr{U}$, and so H_* is open and $H(U_x)=H_*(U)_{H_*(x)}$.

(b) $(K\circ H)[x]=K([H_*(x)])=[K_*\circ H_*(x)]$. Hence $(K\circ H)_*=K_*\circ H_*$.

(c) Since $I(U_x)=U_x$, I_* is the identity.

From theorems 3.1 and 3.4, we can say the homeomorphisms between topological spaces in terms of isomorphisms between their associated semigroups as follows:

THEOREM 3.5. *Two topological spaces are homeomorphic if and only if their corresponding semigroups are isomorphic.*

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