Kyungpook Math. J.
Volume 15, Number 2
December, 1975

# A NOTE ON FINITE *CW*-COMPLEXES

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Let $\mathscr{S}$ be the stable homotopy category (§1) generated by all finite *cw*-complexes. The Grothendieck group $G=K_0(\mathscr{S})$ of $\mathscr{S}$ is defined as follows. For $X,Y\in\mathscr{S}$ we define $X\equiv Y$ if and only if there is a space $W\in\mathscr{S}$ such that $X\vee W\simeq Y\vee W$, where $\vee$ is the wedge operation (§1) and $\simeq$ means to be homotopic. Of course $\equiv$ is an equivalence relation. We put $G=\mathscr{S}/\equiv$, then $G$ is a free abelian group ([1]). In a process of this proof the study of $\Pi_*{}^S(X)$ (§2) for $X\in\mathscr{S}$ is important.

In this paper we shall prove that if $X\underset{Q}{\simeq}Y$ ($Q$-isomorphic) then $\Pi_*{}^S(X)\underset{Q}{\simeq}\Pi_*{}^S(Y)$ and $H_*(\underline{S}X)\underset{Q}{\simeq}H_*(\underline{S}Y)$ in §2 (Theorem 8). For this, it will be proved that for finite *cw*-complexes $X$ and $Y$ $\{X,Y\}$ is a finitely generated abelian group in §1 (Theorem 2). Thus the ring End$X$ of endomorphisms of $X$ is a finitely generated abelian group (§2).

## 1. Stable homotopy category

Let $\mathscr{T}$ be the category consisting of finite *cw*-complexes with base points and maps (continuous maps preserving base points). For $X,Y\in\mathscr{T}$ $[X,Y]$ is the set of homotopy classes $[f]$ of maps $f\colon X\longrightarrow Y$ in $\mathscr{T}$, and $*\colon X\longrightarrow Y$ in the trivial map with $*(x)=*$ for all $x\in X$. We put $0=[*]\in[X,Y]$.

For $X,Y\in\mathscr{T}$, let $X\vee Y=X\times*\cup*\times Y$, the *wedge* of $X$ and $Y$, which is a subspace of $X\times Y$. We define $X\wedge Y=X\times Y/X\vee Y$, the *smash* of $X$ and $Y$.

Let $S\colon\mathscr{T}\longrightarrow\mathscr{T}$ be the *suspension functor*. Since $\wedge$ is distributive over $\vee$, we have for $X\in\mathscr{T}$

$$SX\vee SX=(S\wedge X)\vee(S\wedge X)=(S\vee S)\wedge X.$$

In fact, since $S=[0,1]/\{0,1\}$ ($[0,1]=I$) we can identify $S\vee S$ with $I/\{0,\frac{1}{2},1\}$, and thus we can get the induced pinching map $\nu\colon S\longrightarrow S\vee S$. Hence there is a map $\nu_x=\nu\wedge 1_x\colon SX\longrightarrow SX\vee SX$. If $f_1\colon SX\longrightarrow Y_1$ and $f_2\colon SX\longrightarrow Y_2$ are in $\mathscr{T}$, we define a map

$$(f_1,f_2)=(f_1\vee f_2)\cdot\nu_x\colon SX\longrightarrow Y_1\vee Y_2.$$

If $Y_1=Y=Y_2$ then composing $(f_1,f_2)$ with the natural map $Y\vee Y\longrightarrow Y$ yields $f_1+f_2\colon SX\longrightarrow Y$. With this addition $[SX,Y]$ has a group structure with identity

$0=[*]$.

For $f: X \longrightarrow Y$ in $\mathscr{T}$ we define the *mapping cone* $C_f=(Y \cup I \times X)/\sim$, where $\sim$ is the equivalence relation $(0,x)\sim f(x)$ and $(1,x)\sim(t,*)\sim *$. Then we have the *mapping cone sequence* of $f$

$$X \xrightarrow{f} Y \xrightarrow{i_f} C_f \xrightarrow{\sigma_f} SX \xrightarrow{Sf} SY \xrightarrow{Si_f} SC_f \longrightarrow \cdots\cdots$$

which has the property that every sequence of two maps (and three spaces) is a mapping cone sequence ([1]).

Let $\mathscr{G}\alpha$ be the category of abelian groups. Then there exist the *homology functors* $\{H_n: \mathscr{T} \longrightarrow \mathscr{G}\alpha\}$ satisfying the following properties ([2]):

(i) (Exactness) $H_n(X) \longrightarrow H_n(Y) \longrightarrow H_n(C_f)$ is exact.

(ii) (Connecting) $H_n \cdot S$ is naturally equivalent to $H_{n-1}$.

(iii) (Coefficient) $$H_n(S^m)=\begin{cases} \mathbf{Z} & \text{if } n=m \\ 0 & \text{if } n \neq m, \end{cases}$$

where $\mathbf{Z}$ is the ring of all integers.

(iv) (Hurewicz) There is the natural transformation

$$[S^n, X] \longrightarrow \mathrm{Hom}_{\mathscr{G}\alpha}(H_n(S^n), H_n(X)) \longrightarrow H_n(X)$$

such that if $[S^j, X]=0$ for all $j<n (n>1)$ then $[S^n, X] \longrightarrow H_n(X)$ is an isomorphism. (Note that by the above description $[S^n, X]$ is an abelian group $(n>1)$.)

Let $\mathscr{H}$ be the category of all finite *cw*-complexes with base points and homotopy classes of maps preserving base points. Then $\mathscr{H}$ is a quotient category of $\mathscr{T}$. We now want to extend this category $\mathscr{H}$ to a good category which is called the *stable homotopy category* $\mathscr{S}$.

The category $\mathscr{S}$ is defined as follows. The objects are pairs $(X,n)$ with $(X,n)=(SX,n-1)$ where $X \in \mathscr{H}$ and $n \in \mathbf{Z}$. The morphisms are $\mathscr{S}((X,n),(Y,m))=\varinjlim_r [S^{n+r}X, S^{m+r}Y]=\{S^{n+r}X, S^{m+r}Y\}$, where $r+n,\ r+m>0$. Thus, given a space $X \in \mathscr{H}$ we can refer to its *desuspension* $S^{-1}X$ in $\mathscr{S}$.

Let $\mathscr{G}\alpha^Z$ be the category of graded abelian groups over $\mathbf{Z}$ and degree zero maps. For each $X \in \mathscr{H}$ let $H(X)$ be the *total homology*, i.e, $(H(X))_n=H_n(X)$. Let $\bar{S}: \mathscr{G}\alpha^Z \longrightarrow \mathscr{G}\alpha^Z$ be the shifting automorphism, i.e, for $A=\{A_n\} \in \mathscr{G}\alpha^Z$ $\bar{S}(A_n)=A_{n+1}$. Then the connecting property (ii) on the $H_n$'s gives the commutative diagram

$$\begin{array}{ccc} \mathscr{H} & \xrightarrow{\ S\ } & \mathscr{H} \\ {\scriptstyle H}\downarrow & & \downarrow{\scriptstyle H} \\ \mathscr{G}\mathfrak{a}^Z & \xrightarrow{\ \bar{S}\ } & \mathscr{G}\mathfrak{a}^Z \end{array}$$

Also, there exists a unique $H$: $\mathscr{S} \longrightarrow \mathscr{G}\mathfrak{a}^Z$ still compatable with $S$ and $\bar{S}$ and such that the diagram

$$\begin{array}{ccc} \mathscr{H} & \hookrightarrow & \mathscr{S} \\ & {\scriptstyle H}\searrow \quad \swarrow{\scriptstyle H} & \\ & \mathscr{G}\mathfrak{a}^Z & \end{array}$$

is commutative, where $\hookrightarrow$: $X \longmapsto (X, 0)$ and $S$: $(X, n) \longmapsto (X, n+1)$ ([2]). Note that for $n>0$ $H_{-n}(X)=H_0(S^nX)=H_1(S^{n+1}X)$, that is, $H_n(<X, m>)=H_{n-m}(X)$. Every one of the properties listed for $H_n$: $\mathscr{H} \longrightarrow \mathscr{G}\mathfrak{a}$ holds for the homology functor $H_n$: $\mathscr{S} \longrightarrow \mathscr{G}\mathfrak{a}$, $n \in \boldsymbol{Z}$. For $X \in \mathscr{H}$ if $H_j(X)=0$ for $j \neq n$ and $H_n(X) \cong \boldsymbol{Z} \oplus \cdots \oplus \boldsymbol{Z}$ ($m$-times) then $X$ is isomorphic to $S^n \vee \cdots \vee S^n$ ($m$-times) in $\mathscr{S}$ ([2]).

LEMMA 1. (The 1st stable Dold lemma)

*Let $\mathscr{A}$ be an abelian category, and let $T$: $\mathscr{S} \longrightarrow \mathscr{A}$ be a functor which carries mapping cone sequences into exact sequences. Moreover, let $C$ be a class of objects in $\mathscr{A}$ closed under the formation of kernels, cokernels and exact extensions. That is, if for $A_1, A_2, A_4, A_5 \in C$ $A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow A_4 \longrightarrow A_5$ is exact in $\mathscr{A}$ then $A_3 \in C$.*

*If $TS^n \in C$ for all $n$ then $TX \in C$, $X \in \mathscr{H}$* ([2]).

PROOF. At first, we have to note that for each $(X, n)$ $(=S^nX)\sum_n H_n(<X, n>)$ is a finitely generated abelian group ([2]), because of $X$ is a finite $cw$-complex.

Let $A$ be a class of non-trivial spaces of $\mathscr{S}$ such that $X=(X, 0)$ is in $A$ if and only if $H(X)$ is finitely generated, $H_j(X)=0$ for $j>0$ and $H_0(X)$ is a free $\boldsymbol{Z}$-module. We want to prove that if $X \in A$ then $T(S^nX) \in C$ for all $n$. If we can do that, then for all $X$ with $\sum_n H_n(X)$ finitely generated we have $S^nX \in A$ for some $n$, and thus $T(X) \in C$. Note that if $\sum_n H_n(X)$ is finitely generated then for $j$ sufficientely large $H_j(X)=0$, and that there is $n$ such that $S^nX \in A$. Therefore our lemma is completly proved.

Take $X \in A$. Let $c(X)$ be the smallest integer $j$ such that $H_j(X) \neq 0$. Then $c(X) \leqq 0$. If $c(X)=0$ then $H_n(X)=0$ for $n \neq 0$ and $\mathrm{H}_0(X)$ is free. Thus $X$ is isomorphic to a wedge of $S^0$. By our hypothesis $T(X)=T(\vee S^0)=\oplus T(S^0) \in C$.

Suppose $-c(X)>0$ then for all $j<c(X)$ $H_j(X)=0$. Thus $H_{c(X)}(X)\cong\{S^{c(X)},\ X\}$ (see the above (iv)) is finitely generated. Put $W=$ a wedge of $c(X)$-dimensional spheres such that there is a map $W\longrightarrow X$ which induces the onto homomorphism $\{S^{c(X)},W\}\longrightarrow\{S^{c(X)},X\}$. Consider a mapping cone sequence $W\to X\to Y\to SW\to SX$. Then, for $j>0$ the exact sequence

$$0=H_j(X)\longrightarrow H_j(Y)\longrightarrow H_j(SW)=H_{j+1}(W)=0 \text{ (see (iii) above)}$$

implies that $H_j(Y)=0$. For $j=0$ the exact sequence

$$0\longrightarrow H_0(X)\longrightarrow H_0(Y)\longrightarrow H_0(SW)=H_{-1}(W)\longrightarrow H_0(SX)=H_{-1}(X)$$

implies that $H_0(Y)$ is free, because $H_{-1}(W)\neq 0$ and its subgroups are free. Therefore $Y\in A$. For $j<c(X)$ $0\longrightarrow H_j(Y)\longrightarrow 0$ is exact and for $j=c(X)$ $H_{c(X)}(S^{-1}SW)=H_{c(X)}(W)\twoheadrightarrow H_{c(X)}(X)\longrightarrow H_{c(X)}(Y)\longrightarrow H_{c(X)}(SW)=H_{c(X)-1}(W)$ $=0$ is exact, and thus $H_{c(X)}(Y)=0$. Therefore $c(X)<c(Y)$. Repeating this way we have two sequences of spaces:

$$\{W_1,\cdots,W_n\},\quad \{Y_1,\cdots,Y_n\}$$

such that $W_1\longrightarrow X\longrightarrow Y_1$, $W_2\longrightarrow Y_1\longrightarrow Y_2,\cdots\cdots, W_n\longrightarrow Y_{n-1}\longrightarrow Y_n$ are cofibrations, $W_i(i=1,\cdots,n)$ a wedge of $c(Y_{i-1})$-dimensional spheres with $\{S^{c(Y_{i-1})},\ W_i\}\twoheadrightarrow\{S^{c(Y_{i-1})},Y_{i-1}\}$, and $c(Y_n)=0$. Then $Y_n$ is a wedge of zerodimensional spheres, and thus $T(Y_n)\in C$. Since $T(W_n)\in C$, for all $n$ $T(S^nY_{n-1})\in C$, because of the sequence

$$T(S^{n-1}Y_n)\longrightarrow T(S^nW_n)\longrightarrow T(S^nY_{n-1})\longrightarrow T(S^nY_n)\longrightarrow T(S^{n+1}W_n)$$

is exact and $T(S^{n-1}Y_n)$, $T(S^nW_n)$, $T(S^nY_n), T(S^{n+1}W_n)\in C$. Inductively, we get $T(S^nX)\in C$ for all $n$.

THEOREM 2. *$\mathscr{S}(X,Y)$ is finitely generated.*

PROOF. Let $C$ be the class of finitely generated abelian groups in $\mathscr{G}\mathfrak{a}$. Then $C$ is closed under the formation of kernels, cokernels and exact extensions. For all $n$ the functor $\{S^n,-\}:\mathscr{S}\longrightarrow\mathscr{G}\mathfrak{a}$ with $\{S^n,X\vee Y\}=\{S^n,X\}\oplus\{S^n,Y\}$ ([1]) which carries mapping cone sequences into exact sequences. Since $\{S^n,S^m\}$ is finitely generated $\{S^n,S^m\}\in C$. By the above lemma, for all finite *cw*-complexes $Y$ $\{S^n,Y\}\in C$. The functor $\{-,Y\}:\mathscr{S}\longrightarrow\mathscr{G}\mathfrak{a}$ carries mapping cone sequences into exact sequences, where $Y$ is a finite *cw*-complex. Since $\{S^n,Y\}\in C$ for all $n$, $\{X,Y\}\in C$ for all finite *cw*-complex $X$.

## 2. Some properties of finite *cw*-complexes

The *Freyd category* $\mathscr{F}$, an abelian category, is defined as follows. An object $\alpha$ of $\mathscr{F}$ is a morphism of the stable homotopy category $\mathscr{S}$, i.e., $\alpha \in \{X, Y\}$ for some finite *cw*-complexes $X$ and $Y$. If $\alpha \in \{X, Y\}$ and $\alpha' \in \{X', Y'\}$ then a morphism $m \in \mathscr{F}(\alpha, \alpha')$ is a pair $(m', m'')$ satisfying the commutative diagram

$$\begin{array}{ccc} (X & \xrightarrow{\alpha} & Y) \\ {\scriptstyle m'}\downarrow & & \downarrow{\scriptstyle m''} \\ (X' & \xrightarrow{\alpha'} & Y') \end{array}$$

subject to the identification

$$\begin{array}{ccc} (X & \xrightarrow{\alpha} & Y) \\ {\scriptstyle m'}\downarrow & & \downarrow{\scriptstyle m''} \\ (X' & \xrightarrow{\alpha'} & Y') \end{array} \quad = \quad \begin{array}{ccc} (X & \xrightarrow{\alpha} & Y) \\ {\scriptstyle n'}\downarrow & & \downarrow{\scriptstyle n''} \\ (X' & \xrightarrow{\alpha'} & Y') \end{array}$$

if and only if $m''\alpha = n''\alpha$ (hence if and only if $\alpha'm' = \alpha'n'$). There is a functor $\mu: \mathscr{S} \longrightarrow \mathscr{F}$ with $\mu(X) = (X \xrightarrow{1_X} X)$ and $\mu(f) = (f, f)$. Then $\mu$ is a *full embedding* ([1]). In $\mathscr{F}$ morphisms $(f, 1)$ and $(1, g)$

$$\begin{array}{ccc} (X & \xrightarrow{\alpha} & X') \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle 1} \\ (Y & \xrightarrow{\beta} & X') \end{array}, \qquad \begin{array}{ccc} (X & \xrightarrow{\alpha} & X') \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle g} \\ (X & \xrightarrow{\beta} & Y') \end{array}$$

are a monomorphism and an epimorphism, respectively.

Moreover, given

$$\begin{array}{ccc} (X & \xrightarrow{\alpha} & X') \\ {\scriptstyle f=f'}\downarrow & & \downarrow{\scriptstyle f''} \\ (Y & \xrightarrow{\beta} & Y') \end{array},$$

where $K \xrightarrow{k} X \xrightarrow{f''\alpha} Y'$ is a cofibration, then Ker$f$ in $\mathscr{F}$ is

$$\begin{array}{ccc} (K & \xrightarrow{\alpha k} & X') \\ \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle 1} \\ (X & \xrightarrow{\alpha} & X') \end{array},$$

and given

$$\begin{array}{ccc} (X & \xrightarrow{\alpha} & X') \\ g=g' \big\downarrow & & \big\downarrow g'' \\ (Y & \xrightarrow{\beta} & Y') \end{array},$$

where $X \xrightarrow{g''\alpha} Y \xrightarrow{h} C$ is a cofibration, then Cok$g$ in $\mathscr{F}$ is

$$\begin{array}{ccc} (Y & \xrightarrow{\beta} & Y') \\ 1 \big\downarrow & & \big\downarrow h \\ (Y & \xrightarrow{h\beta} & C) \end{array}.$$

In particular, for a map $f$: $X \longrightarrow Y$ in $\mathscr{T}$ there is the exact sequence

$$0 \longrightarrow \text{Cok} f \longrightarrow C_f \longrightarrow \text{Ker } Sf \longrightarrow 0 \qquad (※)$$

in $\mathscr{F}$, where $X \xrightarrow{f} Y \xrightarrow{i_f} C_f$ is a cofibration, Cok$f = (Y \xrightarrow{i_f} C_f)$, Ker$Sf = (C_f \xrightarrow{\sigma} SX)$ and $C_f = (C_f \xrightarrow{1_{C_f}} C_f)$. The following lemma is well known ([1]).

LEMMA 3. *Every object of $\mathscr{S}$ is $\mathscr{F}$-projective and every $\mathscr{F}$-projective is isomorphic to something in $\mathscr{S}$. Dually everything in $\mathscr{S}$ is $\mathscr{F}$-injective and every $\mathscr{F}$-injective is isomorphic to something in $\mathscr{S}$.*

We need some algebraic prepare for the remainder of this paper. For $X(=(X_1 \xrightarrow{\alpha} X_2)) \in \mathscr{F}$, consider the *ring of endomorphisms* of $X = \text{End } X = \{X, X\}$. We have already proved in Theorem 2 (§1) that End$X$ is a finitely generated abelian group. As is well known, every finitely generated abelian group is the direct sum of a finite number of indecomposable cyclic subgroups, some finite and primary, some infinite. The number of infinite cyclic summands is called the *rank* of the group.

DIFINITION 4. For $X \in \mathscr{F}$, $1_X \in \text{End}X$, where $1_X$ is the class represented by the identity map. If there is an integer $m$ such that $m1_X = 0$, then $X$ is said to be *torsion*. This statement is equivalent to that if End$X$ is finite then $X$ is torsion.

LEMMA 5. *Let $X$ be a finite cw-complex. Then $X$ is torsion if and only if $\Pi_*^S(X)$ is finite in each degree, where* $\Pi_r^S(X) = \varinjlim_n [S^{n+r}, S^n X] = \{S^r, X\}$.

PROOF. If $\Pi_*^S(X)$ is finite in each degree, then it is obvious that $X$ is torsion ([1]). Let $X$ be torsion, then there is an integer $m$ such that $m1_X=0$. Take a generator $f$ of $\Pi_*^S(X)$. Then $m1_X\cdot f=mf=0$, and thus each generator of $\Pi_*^S(X)$ is torsion. Since $\Pi_*^S(X)$ is finitely generated in each degree, $\Pi_*^S(X)$ is finite.

LEMMA 6. *If* $0\longrightarrow W\xrightarrow{f}X\xrightarrow{g}Y\longrightarrow 0$ *is exact in* $\mathscr{F}$, *then* $X$ *is torsion if and only if* $W$ *and* $Y$ *are torsion.*

(Note: Our assumption means that there is the following diagram

$$\begin{array}{ccccc} & & & & 0 \\ & & & & \downarrow \\ & (W_1 & \longrightarrow & W_2) = & W \\ f=f_1 & \downarrow & & \downarrow 1 & \downarrow \\ & (X_1 & \longrightarrow & W_2) = & X \\ g=1 & \downarrow & & \downarrow g_2 & \downarrow \\ & (X_1 & \longrightarrow & Y_2) = & Y \\ & & & & \downarrow \\ & & & & 0 \end{array} \quad \text{(exact)}$$

PROOF. Let $X$ be torsion, then there is an integer $m$ such that $m1_X=0$. Since $m1_X\cdot f=1_X\cdot mf=mf=0=g\cdot m1_X=mg$, we have $f\cdot m1_W=mf\cdot 1_W=0$ and $m1_Y\cdot g=1_Y\cdot mg=0$. So $m1_W=0=m1_Y$ which means that $W$ and $Y$ are torsion. Conversely, let $W$ and $Y$ be torsion. Then there are integers $m$ and $n$ such that $m1_W=0=n1_Y$. Since $g\cdot n1_X=ng\cdot 1_X=n1_Y\cdot g\cdot 1_X=0$ there exists a morphism $h$: $X\longrightarrow W$ with $n1_X=f\cdot h$, $mf\cdot h=f\cdot m1_W\cdot h=0$ implies that $mn1_X=0$. That is, $X$ is torsion.

If there are two maps $f$: $X\longrightarrow Y$ and $g$: $Y\longrightarrow X$ in $\mathscr{T}$ such that for some integer $m$, $gf=m1_X$ and $fg=m1_Y$, then we say that $X$ and $Y$ are *$Q$-isomorphic*, written $X\underset{Q}{\simeq}Y$, where $Q$ is the ring of all rational numbers. Moreover, if there are two maps $f$: $X\longrightarrow Y$ and $g$: $Y\longrightarrow X$ we can get the *canonical homomorphisms* $\mathrm{End}X\rightleftarrows\mathrm{End}Y$ such that for $\xi\in\mathrm{End}X$ $f\xi g\in\mathrm{End}Y$ and for $\sigma\in\mathrm{End}Y$ $g\sigma f\in\mathrm{End}X$.

By our definition $X\underset{Q}{\simeq}Y$ implies that the rank of $\mathrm{End}X$ is the same as one of $\mathrm{End}Y$. Thus if $X\underset{Q}{\simeq}Y$ then

$$\mathrm{End}X\otimes Q \cong \mathrm{End}Y\otimes Q \cong Q\oplus\cdots\oplus Q \ (n\text{-times}),$$

where $n$=the rank of $\mathrm{End}X$ and $\otimes=\otimes_Z$. In particular, if $X$ and $Y$ are torsion then $X\underset{Q}{\simeq}Y\underset{Q}{\simeq}*$.

PROPOSITION 7. *$X \underset{Q}{\simeq} Y$ implies that $\Pi_*^S(X)\otimes Q \cong \Pi_*^S(Y)\otimes Q$ in each degree. Furthermore $X$ is torsion if and only if $\Pi_*^S(X)\otimes Q=0$ in each degree.*

PROOF. The second part is clear by Lemma 5. By our hypothesis there are two maps $f$: $X \longrightarrow Y$ and $g$: $Y \longrightarrow X$ such that $gf=m1_X$ and $fg=m1_Y$.

Let $X$ be torsion, then at least one of $f$ and $g$ is of the finite oder. Therefore $Y$ is also torsion. Thus

$$\Pi_*^S(X)\otimes Q \cong \Pi_*^S(Y)\otimes Q=0.$$

In case the rank of End$X$ is not zero, $f$ and $g$ are generators with infinite order in $\{X, Y\}$ and $\{Y, X\}$, respectively. If $\xi$ is a generator with infinite order in $\Pi_*^S(X)$, then $f\xi$ is a generator with infinite order in $\Pi_*^S(Y)$. Since the converse is true our proof is completed.

In the case $\Pi_*^S(X)\otimes Q \cong \Pi_*^S(Y)\otimes Q$ $\Pi_*^S(X)$ and $\Pi_*^S(Y)$ are said to be *$Q$-isomorphic*, written $\Pi_*^S(X) \underset{Q}{\simeq} \Pi_*^S(Y)$. Since the Hurewicz map $h$: $\Pi_*^S(X) \longrightarrow H_*(\underline{S}X)$ ($H_r(\underline{S}X)=\underset{\overrightarrow{n}}{\lim}\ H_{n+r}(S^n X)$) induces the $Q$-isomorphism ([1]) we have the following.

THEOREM 8. *If $X \underset{Q}{\simeq} Y$ then $\Pi_*^S(X) \underset{Q}{\simeq} \Pi_*^S(Y)$ and $H_*(SX) \underset{Q}{\simeq} H_*(SY)$.*

The converse of this theorem may be not true. But the following holds.

PROPOSITION 9. *If $f$: $X \longrightarrow Y$ induces the $Q$-isomorphism $f_*$: $\Pi_*^S(X) \underset{Q}{\simeq} \Pi_*^S(Y)$, then* Ker$f$ *and* Cok$f$ *are torsion, where $X$ and $Y$ are finite cw-complexes.*

PROOF. From the cofibration $X \xrightarrow{f} Y \xrightarrow{i_f} C_f$ and our assumption we have $\Pi_*^S(C_f)\otimes Q=0$, and thus $\Pi_*^S(C_f)$ is finite. Lemma 5 says that $C_f$ is torsion.

By the above (※) there is the exact sequence

$$0 \longrightarrow \mathrm{Cok}f \longrightarrow C_f \longrightarrow S\mathrm{Ker}f \longrightarrow 0.$$

It follows from Lemma 6 that Cok$f$ and $S$Ker$f$ are torsion. Therefore, also Ker$f$ is torsion.

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