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CONVEXITY THEOREM FOR $[N, p, q]$ SUMMABILITY

By Rajiv Sinha

1. Let $\sum_{n=0}^{\infty} a_n$ be an infinite series, and $\{s_n\}$ be the sequence of its partial sums i.e.,

$$s_n=\sum_{i=0}^{n} a_i .$$

For α real, define

$$A_0^{\alpha}=1,\ A_n^{\alpha}=\frac{(a+1)(\alpha+2)\cdots(\alpha+n)}{n}\quad (n=1,2,\cdots) .$$

Let $\{p_n\}$ be a sequence with $p_0>1$ and $p_n\geq 0$ for $n>0$.
Define

$$p_n^{\alpha}=\sum_{v=0}^{n} A_{n-v}^{\alpha-1}p_v . \tag{1.1}$$

The following identities are immediate:

$$\sum_{v=0}^{n} A_{n-v}^{\beta-1}\, p_v^{\alpha}=p_n^{\alpha+\beta} , \tag{1.2}$$

$$P_n^{\alpha}=p_n^{\alpha+1}=\sum_{v=0}^{n} p_v^{\alpha} , \tag{1.3}$$

where

$$P_n=\sum_{v=0}^{n} p_v .$$

Let $\{q_n\}$ be any sequence of constants, and write

$$(p{*}q)_n=p_0q_n+p_1q_{n-1}+\cdots+p_nq_0 .$$

(N, p^{α}, q) Summability

For $\alpha>-1$ and $\sum_{v=0}^{\infty} a_v$ a series, let

$$t_n^{\alpha}=\frac{1}{(p^{\alpha}{*}q)_n}\sum_{v=0}^{n} p_{n-v}^{\alpha}\, q_v\, s_v . \tag{1.4}$$

If $t_n^{\alpha}\longrightarrow s$ as $n\longrightarrow\infty$ we write

$$\sum_{v=0}^{\infty} a_v = s(N, p^\alpha, q) \text{ or } s_n \longrightarrow s(N, p^\alpha, q).$$

If $t_n^\alpha = 0(1)$ we write

$$\sum_{v=0}^{\infty} a_v \text{ is bounded } (N, p^\alpha, q).$$

$[N, p^{\alpha+1}, q]_\lambda$ **summability**

For $\alpha > -1$, $\lambda > 0$ and $\sum_{v=0}^{\infty} a_v$ a series, we say that $\sum_{v=0}^{\infty} a_v$ is strongly summable $(N, p^{\alpha+1}, q)$ with index λ to s if

$$\frac{1}{(p^{\alpha+1} * q)_n} \sum_{v=0}^{n} (p^\alpha * q)_v |t_v^\alpha - s|^\lambda = o(1)$$

and we write

$$\sum_{v=0}^{\infty} a_v = s[N, p^{\alpha+1}, q]_\lambda \text{ or } s_n \longrightarrow s[N, p^{\alpha+1}, q]_\lambda.$$

We say that $\sum_{v=0}^{\infty} a_v$ is bounded $[N, p^{\alpha+1}, q]_\lambda$ if

$$\frac{1}{(p^{\alpha+1} * q)_n} \sum_{v=0}^{n} (p^{\alpha+1} * q)_v |t_v^\alpha|^\lambda = (1).$$

REMARK. If we take $p_0 = 1$, $p_n = 0$ for $n > 0$ and $q_n = 1$ for $n \geq 0$, then the above definitions yield the standard definition of Cesaro and strong Cesaro summability respectively.

2. In order to prove our theorem, let us restrict the sequence $(p^* q)_n$ by imposing the following condition:

For each $\xi > -1$, there exist positive constants H_1 and H_2 (which may depend on ξ but not on n) such that

$$H_1 n^\xi \leq (p^\xi * q)_n / (p^* q)_n \leq H_2 n^\xi \ . \tag{2.1}$$

The condition (2.1) does not hold good in general.

THEOREM. *If* $\sum_{v=0}^{\infty} a_v$ *is bounded* $[N; p^{\alpha+1}, q]_\lambda$, *and summable* $[N, p^{\beta+1}, q]_\lambda$ *where* $\beta > \alpha > -1$, $\lambda \geq 1$ *and* (2.1) *holds for* $\xi > -1$, *then* $\sum_{v=0}^{\infty} a_v$ *is summable* $[N, p^{\nu+1}, q]_\lambda$ *whenever* $\beta > \nu > \alpha$.

In view of [2, Theorem 5], the above theorem is a consequence of the following lemma.

LEMMA. *If* $\lambda\geq 1$, $\alpha>-1$, (2.1) *holds for* $\xi>-1$ $\sum_{v=0}^{\infty} a_v$ *is bounded* $[N,p^{\alpha+1}, q]_\lambda$ *and summable* $[N,p^{\alpha+2},q]_\lambda$ *to zero, and* $0<\delta<1$, *then* $\sum_{v=0}^{\infty} a_v$ *is summable* $[N, p^{\alpha+\delta+1},q]_\lambda$ *to zero.*

PROOF. We are given that

$$\sum_{n=0}^{m} (p^{\alpha}*q)_n|t_n^{(\alpha)}|^\lambda=o((p^{\alpha+1}*q)_m)$$

and

$$\sum_{n=0}^{m} (p^{\alpha+1}*q)_n|t_n^{(\alpha+1)}|^\lambda=o((p^{\alpha+2}*q)_m).$$

we must prove that

$$\sum_{n=0}^{m} (p^{\alpha+\delta}*q)_n|t_n^{(\alpha+\delta)}|^\lambda=o((p^{\alpha+\delta+1}*q)_m).$$

Now

$$(p^{\alpha+\delta}*q)_n t_n^{(\alpha+\delta)}=\sum_{v=0}^{n} A_{n-v}^{\delta-1}(p^{\alpha}*q)_v t_v^{(\alpha)}$$

$$=\sum_{v=0}^{n-[\theta n]} A_{n-v}^{\delta-1}(p^{\alpha}*q)_v t_v^{(\alpha)}+\sum_{v=n-[\theta n]+1}^{n} A_{n-v}^{\delta-1}(p^{\alpha}*q)_v t_v^{(\alpha)}$$

where θ is any number in the open interval $(0, 1/2)$. Putting $\mu=n-v$ in the second sum and using Abel's partial summation formula on the first sum we obtain

$$\begin{aligned}(p^{\alpha+\delta}*q)_n t_n^{(\alpha+\delta)}= &\sum_{\mu=0}^{[\theta n]-1} A_{\mu}^{\delta-1}(p^{\alpha}*q)_{n-\mu} t_{n-\mu}^{\alpha}\\ &+\sum_{v=0}^{n-[\theta n]-1} A_{n-v}^{\delta-2}(p^{\alpha+1}*q)_v t_v^{(\alpha+1)}\\ &+A_{[\theta n]}^{\delta-1}(p^{\alpha+1}*q)_{n-[\theta n]}\, t_{n-[\theta n]}^{(\alpha+1)}\\ &=U_n+V_n+W_n.\end{aligned}$$

Now

$$\begin{aligned}3^{-\lambda}\sum_{n=0}^{m} (p^{\alpha+\delta}*q)_n|t_n^{(\alpha+\delta)}|^\lambda\leq&\sum_{n=0}^{m}|U_n|^\lambda/\{(p^{\alpha+\delta}*q)_n\}^{\lambda-1}\\ &+\sum_{n=0}^{m}|V_n|^\lambda/\{(p^{\alpha+\delta}*q)_n\}^{\lambda-1}+\sum_{n=0}^{m}|W_n|^\lambda/\{(p^{\alpha+\delta}*q)_n\}^{\lambda-1}\end{aligned}$$

We now consider the three sums on the right hand side of this inequality separately. Using Hölder's inequality we find

$$|U_n|^\lambda \le \Big(\sum_{0\le\mu<\theta n} A_\mu^{\delta-1}(p^\alpha * q)_{n-\mu}|t_{n-\mu}^{(\alpha)}|\Big)^\lambda$$

$$\le \Big(\sum_{0\le\mu<\theta n} A_\mu^{\delta-1}(p^\alpha * q)_{n-\mu}|t_{n-\mu}^{(\alpha)}|^\lambda\Big)\big((p^{\alpha+\delta} * q)_n\big)^{\lambda-1}.$$

Thus

$$\sum_{n=0}^{m}|U_n|^\lambda/((p^{\alpha+\delta}*q)_n)^{\lambda-1} \le \sum_{n=0}^{m}\sum_{0\le\mu<\theta n} A_\mu^{\delta-1}(p^\alpha*q)_{n-\mu}|t_{n-\mu}^{\alpha}|^\lambda$$

$$\le \sum_{0\le\mu<\theta m} A_\mu^{\delta-1}\sum_{(\mu/\theta)<n\le m}(p^\alpha*q)_{n-\mu}|t_{n-\mu}^{(\alpha)}|^\lambda$$

$$= o\Big((p^{\alpha+1}*q)_m \sum_{0\le\mu<\theta m} A_\mu^{\delta-1}\Big)$$

$$= o((p^{\alpha+\delta+1}*q)_m\theta^\delta).$$

So

$$\frac{1}{(p^{\alpha+\delta+1}*q)_m}\sum_{n=0}^{m}|U_n|^\lambda/((p^{\alpha+\delta}*q)_n)^{\lambda-1} = O(\theta^\delta). \tag{3.1}$$

$$|V_n| \le \sum_{v=0}^{n-[\theta n]-1}|A_{n-v}^{\delta-2}|(p^{\alpha+1}*q)_v|t_v^{(\alpha+1)}|$$

$$\le H\sum_{v=0}^{n-[\theta n]-1}(n-v+1)^{\delta-2}(p^{\alpha+1}*q)_v|t_v^{(\alpha+1)}|$$

$$\le H([\theta n]+2)^{\delta-2}\sum_{v=0}^{n}(p^{\alpha+1}*q)_v|t_v^{(\alpha+1)}|$$

$$= o\,[[\theta n]+2)^{\delta-2}(p^{\alpha+2}*q)_n]$$

because if a series is summable $[N,p^{\alpha+2},q]_\lambda$ it is also summable $[N,p^{\alpha+2},q]_1$ for $\lambda>1$. Thus

$$|V_n| = ((p^{\alpha+\delta}*q)_n).$$

Hence

$$\sum_{n=0}^{m}|V_n|^\lambda/[(p^{\alpha+\delta*}p)_n]^{\lambda-1} = \sum_{n=0}^{m}(p^{\alpha+\delta}*q)_n|V_n|_\lambda/[(p^{\alpha+\delta}*q)_n]^\lambda$$

$$= o((p^{\alpha+\delta+1}*q)_m). \tag{3.2}$$

Finally

$$|W_n|/(p^{\alpha+\delta}*q)_n \le H|t_{n-[\theta n]}^{(\alpha+1)}|,$$

so that since $\theta\in(0,1/2)$

$$\sum_{n=0}^{m} |W_n|^{\lambda}/[(p^{\alpha+\delta*}p)_n|^{\lambda} \leq H_1 \sum_{n=0}^{m} |t_n^{(\alpha+1)}|^{\lambda} = o(m).$$

Hence it is easy to see that

$$\sum_{n=0}^{m} |W_n|^{\lambda}/[(p^{\alpha+\delta}*q)_n]^{\lambda-1} = o((p^{\alpha+\delta+1}*q)_m). \tag{3.3}$$

Combining (3.1), (3.2) and (3.3) we find

$$\limsup_{m\to\infty} \frac{1}{(p^{\alpha+\delta+1}*q)_m} \sum_{n=0}^{m} (p^{\alpha+\delta}*q)_n |t_n^{(\alpha+\delta)}|^{\lambda} \leq H\theta^{\delta}.$$

Since θ is any number in the open interval (0, 1/2), it follows that the superior limit above is zero, which yields the desired conclusion.

It may be remarked that for $q_n=1$, $n=0, 1, \cdots$, our theorem reduces to the theorem of Cass [1].

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Kurukshetra University
Kurukshetra, India

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