〈Research Paper〉

Journal of the Korean Institute of Industirial Engineers
Vol. 1 No. 1 June 1975

Optimal Time-sequential Fire Support Strategies

Myungho Oh*

Abstract

The optimal time-sequential distribution of supporting fire against enemy ground units in combat against attacking friendly units is studied. Lanchester type models of warfare are combined with optimal control theory in this investigation. The optimal time-sequential fire-support policy is characterized for a specific problem. Although complete details for the determination of the optimal policy are not given, it is conjectured, on the basis of the theorems which were proved, that for this problem the optimal policy is to always concentrate all supporting fire on the same enemy unit until supporting fire must be lifted.

I. Introduction

The problem of fire support allocation is a problem of interest to the military tactician and planner. In this paper, we will study the problem of artillery fire support allocation against several enemy ground units. The problem is to determine an optimal time-sequential policy for distributing available fire support without wasting it. By wasting of fire support we mean overkilling, i.e., the destruction of enemy forces that does not contribute to the attainment of the friendly objective. Overkill would correspond to a state variable which represents the strength of a ground unit becoming negative.

To determine a good allocation of fire support, one must consider the dynamic nature of combat. Lanchester-type models of warfare(see [15]) have been developed to provide insights into the dynamics of combat. With such a dynamical system model and considering that fire support decisions are made over a period of time, one is led to an optimal-control/differential-game formulation for the determination of optimal time-sequential fire support allocation.

Such a time-sequential combat optimization problems may be cast as either optimal control problems or differential game, depending on whether or not both sides are modeled as rational decision makers. In the exploratory work at hand, we will consider the optimization of combat decision by only one of the combatants. This leads to consideration of an optimal control problem. Such a problem has been referred to as a Lanchester-type optimal control problem by Taylor(see [14]).

The development of solutions to Lanchester-type optimal control problems is not routine matter due to several technical difficulties. Due to physical considerations force levels can not become negative, and we must introduce state-variable inequality constraints (SVIC's), i.e., no state variable (force level) can be allowed to be negative. All Lanchester-type dynamic tactical allocation problems have SVIC's. The maximum principle(in its original form) is inadequate to solve a problem with SVIC's, since an extremal may contain a subarc which

* OR/SA Div., HQ., ROK Army

lies on the boundary of the state space.

Another difficulty for this type of problem is the possible presence of a singular subarc. Consider a control problem in which, for example, a single control variable u appears linearly in the Hamiltonian H. The maximum principle fails to determine the optimal control when $\partial H/\partial u = 0$ for a finite interval of time. The corresponding trajectory is called a singular subarc.

We will study the problem at hand in two phases. In the first phase we will condiser nonsingular extremals, while in the second phase we will consider singular extremals. One result of the study for this problem is that there is no nice algorithm to determine the optimal control for our problem. In general, combat optimization problems dealing with multiple units are extremely complicated, especially if there are some singular subarcs.

Ⅱ. Artillery Fire Support Allocation Model

Consider combat between two ground forces Red and Blue. Each side is composed of two infantry units. Blue infantry units are being supported by artillery fire.

We want to find the best policy for allocating Blue's artillery fire support against Red to achieve the maximum effectiveness. The measure of effectiveness is taken to be the ratio of Blue infantry forces to Red at the given time. The problem is to determine an optimal policy by which the ratio of two infantry forces at the end of the battle can be maximized.

The following assumptions will be made for developing the model.

1. Each infantry unit is only in contact with his opposing unit.

2. The Blue infantry units are moving to contact with the Red units and due to their movement cause negligible attrition to the opposing Red forces. Furthermore, the Red defensive positions are prepared and relatively invulnerable to small arms fire during this approach phase.

3. The defensive Red units cause attrition to the opposing Blue units according to a "square law" attrition process.

4. The Blue artillery delivers area fire against the Red infantry.

For the notational convenience, the Blue forces will be denoted as X_{11} and X_{12} (with corresponding force levels x_{11} (t) and x_{12}(t)) and the Red forces as X_{21} and X_{22}(with corresponding force levels x_{21}(t) and x_{22}(t)).

With the above assumptions we may develop the following combat optimization model:

$$\text{maximize } J = k_1\frac{x_{11}(T)}{x_{21}(T)} + k_2\frac{x_{12}(T)}{x_{22}(T)}$$

$$\begin{aligned} \text{subject to: } \dot{x}_{11}(t) &= -a_1 x_{11}(t) \\ \dot{x}_{12}(t) &= -c_1 x_{12}(t) \\ \dot{x}_{21}(t) &= -a_2 u(t) x_{21}(t) \\ \dot{x}_{22}(t) &= -c_2(1-u(t))x_{22}(t) \end{aligned} \qquad (1a)$$

where $x_{ij}(t)$: force level at time t

$\dot{x}_{ij}(t)$: the time derivatives of state variables x_{ij} (t)

T: time at which the battle terminates

$u(t)$: control variable

a_i, c_i: attrition rate coefficients

k_i: utilities assigned to ratio of survivors.

Since the criterion functional contains force ratios at the terminal time, we may formulate a problem with a state space of lower dimension by defining new state variables as

$$\frac{x_{11}}{x_{21}} = x_1, \quad \frac{x_{12}}{x_{22}} = x_2.$$

Then we can show that (1a) is equivalent to the following optimal control problem:

$$\text{maximize } J = k_1 x_1(T) + k_2 x_2(T)$$

$$\begin{aligned} \text{subject to } \dot{x}_1 &= -a_1 + c_1 u x_1 \\ \dot{x}_2 &= -a_2 + c_2(1-u)x_2 \end{aligned} \qquad (1b)$$

where all constant coefficients remain same as (1a).

Let us assume that we are given the initial conditions

$$x_1(0)=x_1^0,\ x_2(0)=x_2^0.$$

Let us assume also that we are given a specified time T_1 which is called the duration of the combat. We will assume that the battle will be terminated at time T under one of the following conditons:

Condition 1 : $x_1(T)=x_2(T)=0$, for some $T<T_1$.

Condition 2 : $T=T_1$.

Ⅲ. Development of Necessary Conditons of Optimality

To develop the necessary conditons for optimality, let us define the Hamiltonian for the problem (1b),

$$H(t,x,p,\ u)=p_1(-a_1+c_1ux_1)+p_2(-a_2+c_2(1-u)x_2)\ ; \qquad (2)$$

p_1 and p_2 are the adjoint variables, corresponding to the state variables x_1 and x_2, respectively and, satisfying the differential equations:

$$\frac{dp_1}{dt}=-\frac{\partial H}{\partial x_1}=-c_1up_1$$

$$\frac{dp_2}{dt}=-\frac{\partial H}{\partial x_2}=-c_2(1-u)p_2. \quad (3)$$

By Pontryagin's maximum principle, we can show that the nonsingular optimal control will be obtained by maximizing the Hamiltonian over the control variable. Hence we may observe that the extremal control is then given by

$$u(t)=\begin{cases}0 & \text{if } S<0\\ 1 & \text{if } S>0\\ \text{indeterminate} & \text{if } S=0.\end{cases} \quad (4)$$

where $S=c_1p_1x_1-c_2p_2x_2$.

We will call u the extremal control function and S the switching function. The extremal control u so defined may furnish the global maximum, a local maximum, or only a stationary values for the criterion function.

Since the Hamiltonian H is linear function of the control variable u, if the switching function S becomes identically zero for some finite interval of time, the maximum principle fails to provide any information about the optimal control u. In this case all the admissible values of u maximize the Hamiltomian; therefore theory of singular extremals is required to solve this problem. If $S=\partial H/\partial u$ is to be identically zero for some finite interval of time, then all its derivatives with respect to the time must also vanish. That is, the trajectory remains on a singular aubarc if and only if

$$0=\frac{\partial H}{\partial u}=\frac{d}{dt}\left(\frac{\partial H}{\partial u}\right)=\frac{d^2}{dt^2}\left(\frac{\partial H}{\partial u}\right)=\cdots \quad (5)$$

From these conditions, we will have

$$0=\frac{\partial H}{\partial u}=S=c_1p_1x_1-c_2p_2x_2 \quad (6)$$

$$0=\frac{d}{dt}\left(\frac{\partial H}{\partial u}\right)=\dot{S}=-a_1c_1p_1+a_2c_2p_2 \quad (7)$$

$$0=\frac{d}{dt_2}\left(\frac{\partial H}{\partial u}\right)=\ddot{S}=a_1c_1^2p_1u-a_2c_2^2p_2(1-u) \quad (8)$$

From (6)—(8), we have

$$u=\frac{c_2}{c_1+c_2}=s \quad (9)$$

defining s, the singular control.

The generalized Legendre-Clebsch conditon is necessary for singular subarc to yield a maximum return(see [8]).

The condition will be given

$$(-)^k\frac{\partial}{\partial u}\left[\frac{d^{2k}}{dt^{2k}}\left(\frac{\partial H}{\partial u}\right)\right]\leq 0.$$

For the problem at hand,

$$\frac{\partial}{\partial u}\left[\frac{d^2}{dt^2}\left(\frac{\partial H}{\partial u}\right)\right]=a_1c_1^2p_1+a_2c_2^2p_2\geq 0,$$

since $p_i\geq 0$. Therefore, the Legendre-Clebsch condition is satisfied.

For the singular subarc $S=\dot{S}=O$; it follows that

$$\frac{x_1(t)}{x_2(t)}=\frac{a_1}{a_2} \quad (10)$$

we will refer to the surfaces S_x defined by (10), and S_p, defined by (7) as singulr surfaces. Singular arcs lie in these. The surface defined by

$$S=c_1p_1x_1-c_2p_2x_2=0 \qquad (11)$$

is called the switching surface.

Ⅳ. Synthesis of Extremal Control

A. Synthesizing Function

Let X be the Euclidean space on which the state variables x and their associated adjoint variables p are defined, and U be the Euclidean space for control variables. Suppose we can find a real valued function g: $X\to U$ such that $\dot{x}=f(x,g(x,p))$ determines all the extremal trajectories from initial to terminal time. Then the optimal control function may be explicitly determined by the function of state variables and adjoint variables such that the optimal control at each time depends only on the points of the space at which the state variables and the adoint variables are located at the given time. The function $g(x,p)$ will be called a synthesizing function and explicit determination of such a function is called the synthesis of the extremal control, and the explicit determination of the time history of the extremal trajectory is called synthesis of extremal trajectory.

Unfortunately, an existence theorem for the synthesizing function has not been proved. Furthermore, there is no general algorithm for the problem of finding such a function. If the set of terminal values for the state variables is not explicitly specified, it may be extremely difficult to find a synthesizing function. The determination of the synthesizing function must be done on a case-by-case basis. In the problem at hand the terminal state is not specified. However, we know that the control depends upon the state variable x and the adjoint variable p, and it will be determined by the maximum principle when (x,p) does not belong to the singular surface.

The difficulty is that x is only known initially and p is known only at the problem's end until we have managed to match x and p with the maximum principle to get a solution.

However we may analyze the behavior of the switching function, which is a type of synthesizing function. Although it involves both state and adjoint variables we can infer its behavior as a function of time, along an extremal, to establish several results.

B. Synthesis of Control for a Time Interval During Which the Control is Constant

For synthesizing extremal trajectory, let us consider the constant optmial control from the beginning to the end of the battle. Then from the dfferential equation(1b), we will have the following solutions.

For $u=1$;

$$x_1(t)=\max\left[\left(x_1^0-\frac{a_1}{c_1}\right)e^{c_1t}+\frac{a_1}{c_1},\ 0\right]$$

$$x_2(t)=\max\{x_2^0-a_2t,0\} \qquad (12a)$$

$$p_1(t)=k_1e^{c_1(T-t)}$$

$$p_2(t)=k_2 \qquad (12b)$$

$$S(t)=c_1k_1x_1e^{c_1(T-t)}-c_2k_2x_2 \qquad (12c)$$

$$\dot{S}(t)=-a_1c_1k_1e^{c_1(T-t)}+a_2c_2k_2 \qquad (12d)$$

$$\ddot{S}(t)=a_1c_1^2e^{c_1(T-t)} \qquad (12e)$$

For $u=0$.

$$x_1(t)=\max\{x_1^0-a_1t,0\}$$

$$x_2(t)=\max\left\{\left(x_2^0-\frac{a_2}{c_2}\right)e^{c_2t}+\frac{a_2}{c_2},\ 0\right\} \qquad (13a)$$

$$p_1(t)=k_1$$

$$p_2(t)=k_2e^{c_2(T-t)} \qquad (13b)$$

$$S(t)=c_1k_1x_1-c_2k_2x_2e^{c_2(T-t)} \qquad (13c)$$

$$\dot{S}(t)=-a_1c_1k_1+a_2c_2k_2e^{c_2(T-t)} \qquad (13d)$$

$$\ddot{S}(t)=-a_2c_2^2k_2e^{c_2(T-t)} \qquad (13e)$$

For $u=s=\dfrac{c_2}{c_1+c_2}$

$$x_1(t)=\max\left\{\left(x_1^0-\frac{a_1}{\alpha}\right)e^{\alpha t}+\frac{a_1}{\alpha},\ 0\right\}$$

$$x_2(t)=\max\left\{\left(x_2^0-\frac{a_2}{\alpha}\right)e^{\alpha t}+\frac{a_2}{\alpha},\ 0\right\} \qquad (14a)$$

$$p_1(t)=k_1e^{\alpha(T-t)}$$

$$p_2(t)=k_2e^{\alpha(T-t)} \quad (14b)$$

$$S(t)=(c_1k_1x_1-c_2k_2x_2)e^{\alpha(T-t)} \quad (14c)$$

$$\dot{S}(t)=(-a_1c_1k_1-a_2c_2k_2)e^{\alpha(T-t)} \quad (14d)$$

$$\ddot{S}(t)=\frac{c_2}{c_1+c_2}e^{\alpha(T-t)}(a_1c_1{}^2k_1-a_2c_2k_2) \quad (14e)$$

where

$$\alpha=\frac{c_1c_2}{c_1+c_2}$$

We will use the above results to determine the optimal policy for more general cases.

C. Development of Theorems on the Characterization of an Optimal Policy

By analyzing the behavior of switching function, we may establish the following theorems.

THEOREM 1

Let E be an maximizing extremal which contains no singular subarc for the optimal control problem given by (1b). Then there is at most one switching in E.

Proof:

By hypothesis, since E does not contain a singular subarc, the optimal control can be determined by Pontryagin's maximum principle, except possibly at one point, as we shall see. We have got the switching function from the Hamiltonian by means of the maximum principle,

$$S(t)=c_1p_1x_1-c_2p_2x_2.$$

Then its derivative is

$$\dot{S}(t)=a_2c_2p_2-a_1c_1p_1$$

and $\ddot{S}(t)=a_1c_1{}^2p_1u-a_2c_2{}^2p_2(1-u)$.

It follows that

$$\ddot{S}(t)>0 \text{ if } u=1$$

$$\ddot{S}(t)<0 \text{ if } u=0.$$

Since the trajectory does not lie on singular subarc, we will have either $u=1$ or $u=0$ at $t=0$. Let us consider the case $u=1$ at $t=0$; a similar discussion will hold if $u=0$ initially. Since $u=1$ at $t=0$, $S(0)\geq 0$.

1. Let us consider first the cases in which $S(0)>0$. There are two sub-cases, accordingly as $\dot{S}(0)\geq 0$ or $\dot{S}(0)<0$.

Case 1: $S(0)>0$ and $\dot{S}(0)\geq 0$.

Since $S(0)>0$, we must choose $u=1$ at the beginning of the game. From(12b), $p_1(t)=k_1e^{\alpha(T-t)}$, $p_2(t)=k_2$. Therefore, $p_1(t)$ is decreasing function of t and $p_2(t)$ is constant. Hence $\dot{S}(0)\geq 0$ implies $\dot{S}(0)\geq 0$ and $S(t)>0$, for all $t\geq 0$. Therefore $S(t)$ is monotone increasing function, and consequently we will have $S(t)>0$ for $t\in[0,T]$ Hence no switching occurs.

Case 2: $S(0)>0$ and $(0)\dot{S}<0$.

From equation(12c), S must have the form

$$D(t)=-Ae^{-c_1t}+B \quad (15)$$

in which A and B are positive, so long as $u=1$. But if S becomes negative u must become 0 and S will have a different form.

Let τ be the value of t for which $D=0$. Consider also S: S may become zero at some value $t_1<\tau$ or $S(\tau)$ may be greater than or equal to zero. There are three subcase.

Subcase i

It may be that $S>0$ for all $t\leq\tau$. In this case $S>0$ for all t, since $\ddot{S}>0$ when $u=1$. Hence no switching occurs.

Subcase ii

It may be that S becomes zero at some time $t_1<\tau$ and $\dot{S}(t_1)<0$. Then there is switching at $t=t_1$. From the facts that $S(t_1)=0$, $\dot{S}(t_1)<0$ and $\ddot{S}<0$ when $u=0$, it follows that t_1 is switching point, and that $S<0$ for all $t>t_1$.

Subcase iii

It may be that t_1 in case ii coincides with τ, so that $S(t_1)=0$ and $\dot{S}(t_1)=0$ for some $t_1<T$. Then the maximum principle does not determine the control at time t_1. If we choose $u=1$ at $t=t_1$, when $\ddot{S}(t_1)>0$. If follows that $S(t)$, $\dot{S}(t)$ are positive for $t>t_1$ and there is no switching. If we choose $u=0$ at $t=t_1$, then by the above arguments, $S(t)<0$ for $t>t_1$ and there is one switching.

2. It may be that $S(0)=0$. If $\dot{S}(0)>0$.

then $S(t)$ is always positive. If $\dot{S}(0)<0$ then $S(t)<0$ for $t>0$ and $u\equiv0$. Finally if $\dot{S}(0)=0$ then we may choose $u(0)$ arbitrarily. If we choose $u=1$ initially then $S>0$ for $t>0$ by argument given in case 2-iii and there is no switching. Similarly, if we choose $u(0)=0$, S remains negative.

This takes care of all extremals which do not have a singular arc; a nonsingular extremal has at most one switching point.

Furtheremore, we may develop the following theorems which will provide the informations of extremal control. We will not prove these theorems in this paper. The interested reader may refer author's paper(see [10])

THEOREM 2

If E contains a singular subarc, then there are at most two switchings.

THEOREM 3

Consider a path E_{10} defined by the constant control $u=1$, $0\leq t<t_1$, and $u=0$, $t_1\leq t\leq T$, so that E_{10} has one switching and no singular subarc. Let J_{10} be the associated criterion function. If $x_1^0\geq a_1/c_1$ and $x_2^0\geq a_2T$, then there is no maximizing extremal of the form E_{10}. By symmetry, there cannot be one of the form E_{01} in which $x_2^0\geq a_2/c_2$ and $x_1^0\geq a_1T$. That is, a properly chosen constant control is always better than a mixed policy of this type.

THEOREM 4

All nonsingulr controls which maximize the criterion function for this problem have the form $u\equiv1$ or else $u\equiv0$.

V. Determination of the Optimal Control

A. Sufficient Conditions for Optimality

As is well known the maximum principle only provides necessary conditions of optimality. Second order sufficient conditions(see [1]) are impractical to apply to the problem at hand. Mangasarian's sufficient conditions (see [9], unfortunately do not apply to the problem at hand, since the right-hand sides of the differential equation for the problem are not concave functions of the state and control variables. We will develop some theorems to determine the optimal control for this particular problem.

B. Determination of Optimal Control in Terms of Initial Conditions and Terminal Conditons

In general we cannot determine the optimal control without some computation and comparison. However, in some cases we can establish its form easily. In this section two theorems are developed. The first one establishes a sufficient conditon for some nonsingular arcs to be optimal. The second theorem establishes a necessary condition: An optim iz ing arc never has a terminal singular subarc.

THEOREM 5

If E does not contain a singular subarc, and if $x_1\geq\frac{a_1}{c_1}$ and $0<T<\min\left\{\frac{x_2^0}{a_2},\ \frac{1}{c_2}\ln\frac{c_1k_1x_1^0}{c_2k_2x_2^0}\right\}$, then constant control $u=1$ is the optimal control. By symmetry, if $x_2\geq\frac{a_2}{c_2}$ and $0<T<\min\left\{\frac{x_1^0}{a_1},\ \frac{1}{c_1}\ln\frac{c_2k_2x_2^0}{c_1k_1x_1^0}\right\}$, then the constant control $u=0$ is the optimal control.

Proof:

See[10].

THEOREM 6

A maximizing control does not have a singular subarc ending at $t=T$.

Proof:

See[10].

Thus we see that if a trajectory ends with a singular subarc at the end of the battle, it cannot be an optimal trajectory.

From studying this problem we feel that following result is true. If so it would reduce this particular problem to the two cases for the optimum, i.e., $u \equiv 0$ or $u \equiv 1$. We were unable to prove this result or to find a counter example.

THEOREM 7

A maximizing arc never has a singular subarc nor a corner; that is the optimal control is always either $u \equiv 1$ or $u \equiv 0$, depending on the given conditions.

Ⅵ. Discussion

Considering the theorems given above, we see that if an extremal does not contain a singular subarc, the optimal control u is (a) constant for all $t \in (t, T)$ and is (b) either 0 or 1. Which of these policies is better may be determined by direct computaion and comparison of values of the criterion functional using the attrition rate coefficients and the initial conditons. Thus, we see that in this case the optimal time-sequential fire-support policy for Blue is to concentrate all his artillery resources on one of the Red units for the entire duration of battle. It should be pointed out that even if an extremal contains a singulr subarc, Blue's artillery fire will (optimally) still be concentrated on one of the Red units at the end of battle. This is because an optimal fire-support policy has been characterized as not terminating with a singular subarc.

We have not been able to show which policy is optimal when the trajectory contains a singular subarc for some period of time before the end of the problem. Thus, the complete explicit determination of the optimal fire support policy was not accomplished. However, the basis has been given for doing this in the future.

It should be pointed out that the insights gained into optimal fire-support policies obt ained from the model (1) are no more valid than the model itself. Thus the reader should be cautioned that the optimal fire-support policy determined here may only apply to the problem(1). As discussed by Taylor(see [14]), a combat optimization problem consists of the following three parts: (1) decision criteria, (2) model of the planning horizon (e.g. conflict termination condition), and(3) model of combat dynamics, the optimal fire-support policy may well be different from that given here.

Ⅶ. Conclusions

For the problem considered in this paper, the optimal fire-support policy is piecewise constant with at most two switches. If one never divides fire between the two enemy units (singular control), then the optimal policy is constant and either 0 or 1(i.e., concentrate all fire on one of the enemy units). Although complete details could not be worked out, it is conjectured that this is the optimal policy (i.e., always concentrate all supporting fire on one of the enemy units). The reader should be cautioned that these conclusions may only apply to the specific fire support problem studied here.

BIBLIOGRAPHY

[1] Bryson A.E. and Ho Y.C., *Applied Optimal Control*, Blaisdell Publishing Co., Waltham, Massachusetts, 1969.

[2] Ewing G.M., *Calculus of Variations with Applications*, W.E. Norton and Co. Inc., New York, 1969.

[3] Faulkner F.D., *Optimum Ship Routing*, Naval Postgraduate School, 1962.

[4] Issacs R., *Differential Games*, John Wiley and Sons, Inc., New York, 1965.

[5] Jacobson D.H. and Speyer J.L., "Necessary and Sufficient Conditions for Optimality for Singular Control Proble," *J. Math. Anal. Appl.* 34, 239-266, 1971.

[6] Jacobson, D.H. and Lele, M.M. "Necessary Condition of Optimality for Control Problems with State-Variable Inequality Constraints," *J. Mathn. Aal. App.* 35, 255-284, 1971.

[7] Johnson C.D. and Gibson, J.E. "Singular Solution in Problems of Optimal Control,"

IEEE, Transactions on Automatic Control, 4-15, 1963.

〔8〕 Kelley, H.J. Kopp, R.E. and Moyer, H.G. "Singular Extremals," in Topics in *Optimization*, G. Leitman (ed.), Academic Press, New York, 1967, pp. 63-101.

〔9〕 Mangasarian, O.L. "Sufficient Conditions for the Optimal Control of Nonlinear Systems," *J. SIAM Control* Vol. 4, No. 1, 1966.

〔10〕 Oh, M. *Optimal time-sequential Distribution of Supporting Fire*, Naval Postgraduate School, 1974.

〔11〕 Pontryagin, L.S. Boltyanskii, V.G. Gamkrelidze, R.V. and Mischemo, E.F. *The Mathematical Theory of Optimal Processes*, John Wiley and Sons, Inc., New York, 1962.

〔12〕 Pugh G.E. and Mayberry, J.P. *Analysis of General Purpose Forces Procurements*, Lambda Corporation, paper 59, August 1971.

〔13〕 Taylor, J.G. "Target Selection in Lanchester Combat: Linear-Law Attrition Process," *Naval Research Logistics Quarterly*, Vol. 20, No. 4, pp. 673-697, 1973.

〔14〕 Taylor, J.G. "Survey on the Optimal Control of Lanchester-Type Attrition Processes," to appear in Proceedings of *The Conference on the State-of-the-Art of Mathematics in Combat Model*.

〔15〕 Weiss, H.K. "Lanchester-Type Models of War fare," Proceedings of *the First International Conference Operations Research*, Oxford September 1957.