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Preventive Maintenance Scheduling Guide

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Abstract

This paper presents a generalized model for determining minimum cost preventive maintenance schedules where accurate failure data are not available except the "average"(mean) and the "typical" value(mode) of the component lifetime.

A study[1] of maintenance operations of a major manufacturer in the Detroit area reports that: (1) About 82% of the total yearly downtime is due to unscheduled repairs of machines, tools and dies; (2) No efficient data collection system is in existence to record failures of parts and/or machines. The same study reports that in some cases, where an elementary data collection system is operational, the sketchy but valuable historical data are periodically destroyed as part of profit improvement programs.

Generally, the lack of failure history on machines is a major obstacle in the development of an efficient maintenance scheduling system.

Preventive Maintenance of Failing Units

In this paper the term "maintenance" of a failing unit means a corrective action which may consist of routine adjustments or replacements of sub-components performed on the unit in order to bring the unit to its intended performance level.

The policy suggested here is the age replacement model developed by Barlow[2] in which a unit is maintained T hours after its previous maintenance or at failure, whichever occurs first. An optimal maintenance policy should balance the cost of failures of a unit during operation against the cost of planned maintenance.

An average cost of regularly scheduled preventive maintenance is $\$_p$. An average cost of $\$_f$ is suffered for each in-service failure; this includes all costs resulting from the failure (e.g., the costs of downtime and possible lost sales, idle direct and indirect labor, delays in dependent processes and increased scrap), as well as the costs of repair.

Failure Distribution

When accurate failure data are not available, a Gamma distribution can be used in a wide variety of situations. A Gamma distribution

$$f(t)=\frac{\lambda}{\Gamma(\alpha)}(\lambda t)^{\alpha-1}e^{-\lambda t}$$

where t=age of the unit≥ 0

α=shape parameter≥ 0

λ=scale parameter

has the following general characteristics:

1) Most of the empirical distributions(including the truncated normal distribution) can be represented, at least roughly, by suitable choice of the parameters.

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2) It has increasing failure rate. Thus the life distribution of any structure can be described adequately which, when in normal use, undergoes changes affecting its future life length.
3) The distribution has a "typical" value (mode) which may not be at $t=0$. This characteristic is not shared by an exponential distribution even if the latter has many desirable mathematical properties.
4) The distribution is skewed to the right and therefore seems more natural to describe failure phenomenon which is defined only on the positive time axis.
5) Most importantly, estimates of the unit's "average" lifetime and the "typical" value of the lifetime are sufficient to describe the specific failure distribution. That is, $\lambda=1/(\text{mean-mode})$ and $\alpha=\lambda\text{-mean}$

where λ, α = scale and shape parameter, respectively

mean = average lifetime of the unit > mode

mode = typical lifetime of the unit. (See Appendix.)

Scheduling Model

Under the maintenance policy described earlier, the expected cost of operation per unit time (with the preventive maintenance at age T), ₮(T), is

₮(T) = Exp (cost per maintenance)/Exp (inter-maintenance time)

$$=\frac{\$_f\cdot F_\alpha(T)+\$_p\cdot\bar{F}_\alpha(T)}{\int_0^T xf(x)dx+T\cdot\bar{F}_\alpha(T)}$$

$$=\frac{\$_f+(\$_p-\$_f)\bar{F}_\alpha(T)}{\int_0^T\bar{F}_\alpha(x)dx} \quad \text{(Eq. 1)}$$

where $F_\alpha(t)=1-\bar{F}_\alpha(t)$ = cumulative distribution function of $f(t)$ with shape parameter α.

Unfortunately, there is no simple closed-form expression for the $\bar{F}_\alpha(t)$, especially when the shape parameter α is small. However, when α is an integer a, it can be shown that (4)

$$\bar{F}_\alpha(T)=\sum_{n=0}^{a-1}\frac{e^{-\lambda T}(\lambda T)^n}{n!}$$

Thus it is preferable to interpolate numerically between solutions for integral a. To perform this interpolation, let's denote the largest integer which is less than or equal to the shape parameter α by a. That is,

$$a=[\alpha]^{\text{integer}}\leq\alpha$$

Then

$$\bar{F}_\alpha(T)=(\alpha-a)e^{-\lambda T}(\lambda T)^a/a!+\sum_{n=0}^{a-1}e^{-\lambda T}(\lambda T)^n/n!$$

and $$\int_0^T\bar{F}_\alpha(t)dt=\frac{\alpha}{\lambda}-\frac{(\alpha-a)}{\lambda}\sum_{m=0}^{a}\frac{e^{-\lambda T}(\lambda T)^m}{m!}-\frac{1}{\lambda}\left\{\sum_{n=0}^{a-1}\sum_{m=0}^{n}\frac{e^{-\lambda T}(\lambda T)^m}{m!}\right\}$$

(See Appendix.)

That, a numerically solvable expression for ₮(T) is,

$$₮(T)=\frac{\$_f+(\$_p-\$_f)\left\{(\alpha-a)P_a+\sum_{n=0}^{a-1}P_n\right\}}{\frac{\alpha}{\lambda}-\frac{(\alpha-a)}{\lambda}\sum_{m=0}^{a}P_m-\frac{1}{\lambda}\sum_{n=0}^{a-1}\sum_{m=0}^{n}P_m} \quad \text{(Eq. 2)}$$

where

$$P_n=e^{-\lambda T}(\lambda T)^n/n!$$

Scheduling Procedure Summary

The following are the basic procedures for scheduling of a preventive maintenance program.

1. The decision maker wishes to select a convenient single maintenance interval T^*.
2. The decision maker estimates the "average" lifetime and the "typical" lifetime of the unit. From these, $\lambda=1/(\text{mean}-\text{mode})$, $\alpha=\lambda\cdot\text{mean}$, and a=integer value of α.
3. The decision maker chooses a convenient maintenance interval T^* which minimizes ₮(T) in Equation 2.
4. As data on unit usage and failure accu-

mulates, he can update his estimates of the "average" lifetime and the "typical" lifetime.

5. It can also be shown that $\mathfrak{T}(T)$ is a unimodal function. Thus an elementary numerical search technique can be used to find the exact optimum maintenance interval T^* if desired.

Illustrative Example

Suppose a vital part of a machine has a typical lifetime of 9 months and an average lifetime of 12 months. Then $\lambda=1/(12-9)=1/3$ and $\alpha=12\cdot1/3=4$. If $\$_p=\$10$ and $\$_f=\$50$, the resulting cost of operation for several trial values of preventive maintenance interval can be computed from Equation 2. The values of preventive maintenance interval and the corresponding cost of operation per month is shown in Figure 1("true cost" curve). Although the optimum preventive maintenance interval is 5.9 months with corresponding minimum cost of operation \$2.72/month, the cost curve is shallow around the optimum value. That is, +10%(−10%) error in computing the optimum preventive maintenance interval gives only 0.63%(0.66%) increase in the resulting cost of operation.

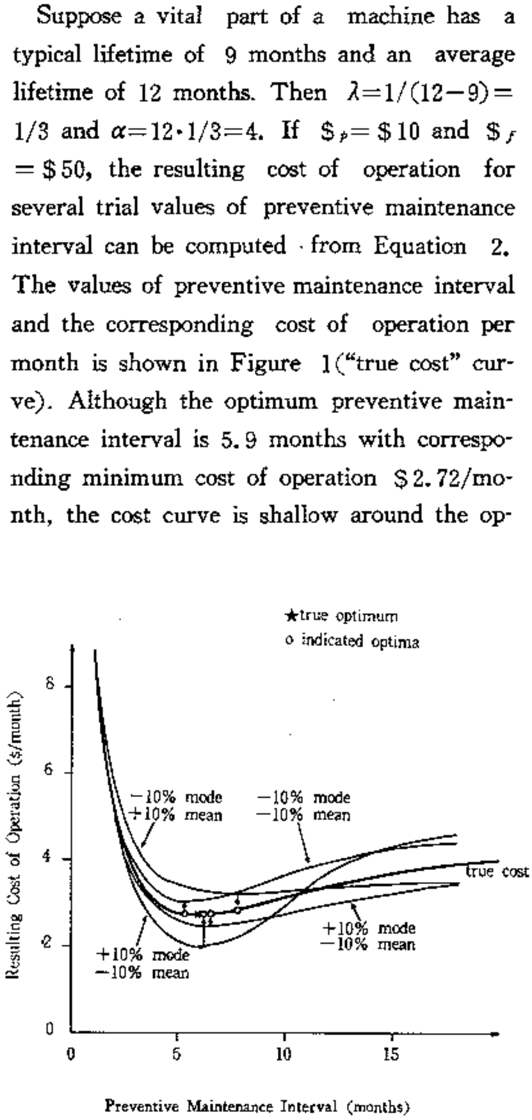

Figure 1

Sensitivity of Cost of Operation to Errors in Mode and Mean

Sensitivity Analysis

In the previous example, suppose that the decision maker makes ±10% errors in estimating the typical lifetime and the average lifetime. The apparent costs of operation under the error conditions are depicted in Figure 1 with their indicated optimum preventive maintenance intervals marked by small circles. If the decision maker uses these indicated optimum preventive maintenance intervals, the resulting true cost of operation would be slightly higher than the true optimum. Table 1 summarizes the percentage increase in the resulting cost of operation. Again, the increase in the resulting cost of operation is very slight.

Table 1 Sensitivity to errors in estimating mode and mean

Error Condition		Indicated Optimum preventive maintenance interval (month)	Corresponding true cost of operation /month	% increase in the cost of operation
mode	mean			
0%	0%	5.9	\$2.7206	0%
−10%	−10%	5.3	\$2.7394	0.69%
−10%	+10%	7.8	\$2.8509	4.8 %
+10%	−10%	6.2	\$2.7258	0.19%
+10%	+10%	6.5	\$2.7384	0.65%

Sensitivity to the Underlying Distribution

The model developed in this paper is insensitive to the moderate departures of the underlying lifetime distribution. For numerical illustration, let's assume that $\$_p=\$10$ and $\$_f=\$50$. Suppose a vital part of a machine has a Weibull lifetime distribution: $f(t)=2te^{-t^2}$(t is in years). Without loss of generality, it is assumed that its scale parameter is 1.

The Weibull distribution has been widely used to describe component failures. It is perhaps the most popular parametric family of failure distributions at the present time.

Since $\bar{F}(T)=e^{-T^2}$ for the given distribution, Equation I becomes

$$\Psi(T)=(50-40e^{-T^2})\sqrt{\pi}[\Phi(\sqrt{2}\cdot T)-0.5]$$

Where $\Phi[\cdot]$ denotes the cumulative unit normal distribution.

The values of preventive maintenance interval and the corresponding true cost of operation per year is shown in Figure 2 ("true cost") when the underlying distribution is, in fact, Weibull. The true optimum preventive maintenance interval is .511 years with corresponding minimum cost of operation $40.85 per year.

However, if the true distribution is not known except its true mean $\left(=\int_{\infty}^{0}2te^{-t^2}dt=\sqrt{\pi}/2\right)$ and its true mode($=1/\sqrt{2}$ which is obtained by setting the derivative of $2te^{-t^2}$ equal to 0, the decision maker would assume a Gamma distribution with $\lambda=2/(\sqrt{\pi}-\sqrt{2})$ and $\alpha=\sqrt{\pi}/(\sqrt{\pi}-\sqrt{2})$. The values of preventive maintenance interval and cost of operation per year under the Gamma approximation are shown in Figure 2("apparent cost").

This "apparent cost" curve indicates that the optimum preventive maintenance interval is .44 years. The "true cost" of operation per year is $41.24 if this indicated preventive maintenance interval is used. The resulting increase in the cost of operation is only 0.95%.

Conclusion: Use of Scheduling Chart

The preventive maintenance scheduling model presented in this paper is not sensitive to the underlying lifetime distributions and to the errors in estimating the "typical" and

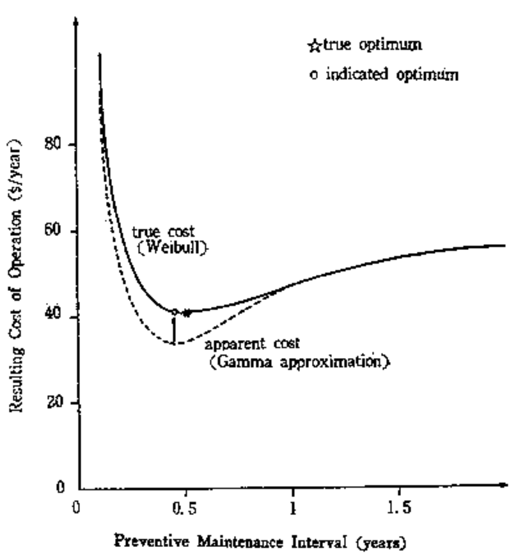

Figure 2

Sensitivity of Cost of Operation to Type of Failure Distribution

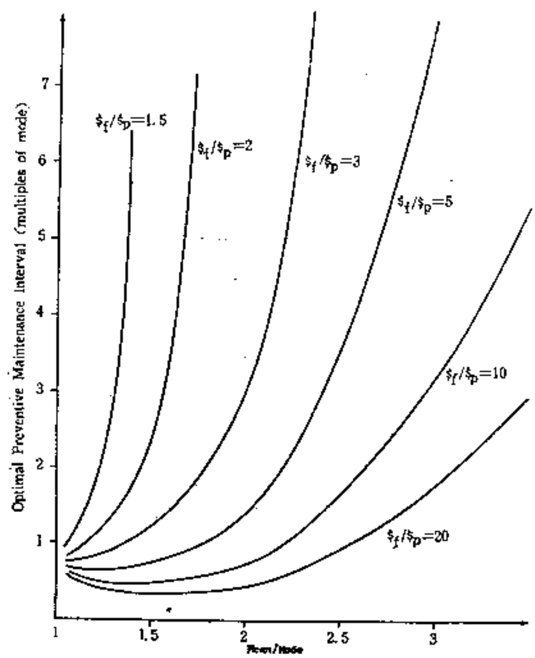

Figure 3

Preventive Maintenance Scheduling Chart

"average" lifetime. The model can be used under a wide variety of conditions when the average lifetime is longer than the typical lifetime, which is a realistic description of the general failure behavior of components.

Figure 3 is a summary of minimization of Equation 2 for different values of mode, mean, $\$_p$, and $\$_f$. This chart can be used effectively in place of Equation 2 to plan a preventive maintenance schedule.

Note, however, that a preventive maintenance policy is economically of no value if mean/mode$\geq \$_f/\$_p$. (See Appendix.)

Appendix

The Gamma distribution is defined by the equation

$$f(t)=\frac{\lambda}{\Gamma(\alpha)}(\lambda t)^{\alpha-1}e^{-\lambda t},\ \ t\geq 0$$

Its mean is $\int_0^\infty f(t)dt=\alpha/\lambda$ and the mode is obtained by setting the derivative of $f(t)$ equal to 0:

$$\frac{d}{dt}f(t)=\frac{\lambda}{\Gamma(\alpha)}\lambda e^{-\lambda t}(\lambda t)^{\alpha-2}\{(\alpha-1)-\lambda t\}=0,$$

or, mode$=(\alpha-1)/\lambda$.

Solving simultaneously, $\lambda=1/$(mean-mode) and $\alpha=$mean/(mean-mode)$=\lambda\cdot$mean.

Furthermore, since

$$\bar{F}_a(T)=\sum_{n=0}^{a-1}e^{-\lambda T}(\lambda T)^n/n!,\ \ \text{for } a\leq\alpha\leq a+1,$$

$$\bar{F}_\alpha(T)\fallingdotseq(\alpha-a)\bar{F}_{a+1}(T)+(a+1-\alpha)\bar{F}_a(T)=(\alpha-a)e^{-\lambda T}(\lambda T)^a/a!+\sum_{n=0}^{a-1}e^{-\lambda T}(\lambda T)^n/n!$$

Then,

$$\int_0^T\bar{F}_\alpha(t)dt=(\alpha-a)\int_0^T\frac{e^{-\lambda t}(\lambda t)^a}{a!}dt+\sum_{n=0}^{a-1}\int_0^T\frac{e^{-\lambda t}(\lambda t)^n}{n!}dt$$

$$=\frac{(\alpha-a)}{\lambda}\left\{\sum_{m=a+1}^{\infty}\frac{e^{-\lambda T}(\lambda T)^m}{m!}\right\}+\frac{1}{\lambda}\sum_{n=0}^{a-1}\sum_{m=n+1}^{\infty}\frac{e^{-\lambda T}(\lambda T)^m}{m!}=\frac{\alpha}{\lambda}$$

$$-\frac{(\alpha-a)}{\lambda}\left\{\sum_{m=0}^{a}\frac{e^{-\lambda T}(\lambda T)^m}{m!}\right\}-\frac{1}{\lambda}\sum_{n=0}^{a-1}\sum_{m=0}^{n}\frac{e^{-\lambda T}(\lambda T)^m}{m!}$$

which follows from repeated integration by parts(3).

THEOREM: Let the random time to failure of a unit have a Gamma distribution. Then if

$$\frac{\text{mean}}{\text{mode}}\geq\frac{\$_f}{\$_p}$$

the optimal policy is not to perform preventive maintenance at all. Otherwise, the optimal preventive maintenance interval T^* will be a finite number.

PROOF OF THEOREM: Let the monotonically increasing function $h(t)$ denote the failure rate of the Gamma probability density function, $f(t)$: $h(t)$

$$=\frac{(\lambda t)^{\alpha-1}e^{-\lambda t}}{\int_t^\infty(\lambda x)^{\alpha-1}e^{-\lambda x}dx}$$

The optimum preventive maintenance inteaval, T^*, is the solution of the equation

$$\frac{d}{dT}\Psi(T)=[(\$_f-\$_p)h(T)\cdot\int_0^T\bar{F}_\alpha(x)dx-\{\$_p+(\$_f-\$_p)F_\alpha(T)\}]\Big/\left[\left\{\int_0^T\bar{F}_\alpha(x)dx\right\}^2/\bar{F}_\alpha(T)\right]=0$$

or,

$$h(T)\int_0^T\bar{F}_\alpha(x)dx-F_\alpha(T)=\frac{\$_p}{\$_f-\$_p}$$

Let $S(T)$ denote the left-hand side of the above equation.

Since $\frac{d}{dT}S(T)=\frac{dh(T)}{dT}\cdot\int_0^T\bar{F}_\alpha(x)dx>0$ for $T>0$, the $S(T)$ is also a monotonically increasing function. Therefore, in order for the T^* to be finite,

$$[S(T)]_{T=\infty}>\frac{\$_p}{\$_f-\$_p}$$

But for a Gamma distribution, if

$$\frac{\text{mean}}{\text{mode}}=\frac{\alpha}{\lambda}\Big/\frac{\alpha-1}{\lambda}=\alpha/\alpha-1\geq\frac{\$_f}{\$_p}$$

$$\text{or } \alpha \leq \frac{\$_f}{\$_f - \$_p},$$

$$[S(T)]_{T=\infty} = \lambda \int_0^\infty \bar{F}_\alpha(x)\,dx - [F_\alpha(T)]_{T=\infty}$$

$$= \lambda \cdot \text{mean} - 1 = \alpha - 1$$

$$\leq \frac{\$_f}{\$_f - \$_p} - 1 = \frac{\$_p}{\$_f - \$_p}$$

since $[h(T)]_{T=\infty} = \lambda$. (Q.E.D.)

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