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Test of Independence in a Markov Dependent Waiting-time Distribution

Do Sun Bai*

Abstract

A procedure for the test of independence of the observations and the null distribution are studied for a waiting-time distribution of the number of Bernoulli trials required to obtain a preassigned number of successes under Markov dependence. Selected critical values for the test statistic are tabulated.

1. Introduction.

A waiting-time distribution of the number of Bernoulli trials required to give a preassigned number of successes under Markov dependence has many useful applications such as in the design of a new weapon system or digital communications system.

In a tank weapon system, for example, the number of rounds to destroy a target may be taken to describe one of its lethality characteristics. The number of rounds (trials) to obtain a preassigned number of hits (successes) so as to destroy a target is important in the study of weapons of this type. Since in a tank gun different adjustments are needed depending upon the results of the previous rounds, the firing process, that is, the sequence of hits and misses may be described by a simple two-state Markov chain with stationary transition probabilities.

The distribution depends on three parameters; the conditional probability α of obtaining a success given that a success occurred on the previous trial, the conditional probability β of a success given that the previous trial resulted in a failure, and the probability p of success on the first trial.

The distribution has first been studied by Bonder(1967) in connection with the determination of Lanchester attrition-rate coefficient. Rustagi and Srivastava (1968) considered the problem of estimating these parameters, and have shown that when the total number of trials required and the order of successes and failures are observable, there is a set of sufficient statistics for the parameters (p,α,β).

In such case, the maximum likelihood estimates can easily be obtained. Rustagi and Laitinen(1970) considered the case when only the total number of trials required is observable and obtained moment estimates. Laitinen and Rustagi(1972) gave the asymtotic distribution of these moment estimates and showed that they are regular best asymtotically normal. Bai and Rustagi(1974) considered the problem of minimum variance unbaised estimation of the parameters α and β under the condition that only the total number of trials required, the outcome of the first trial and the number of runs of successes are observable and have shown that the maximum likelihood estimate of α is minimum variance unbaised while that of β tends to over-estimate the true value. They proposed an alternative estimate for β

* Department of Industrial Engineering
Korea Advanced Institute of Science

that minimizes the bias. These estimates were found to be independent of each other and have asymtotic normal distributions. They also suggested a procedure for testing independence of the trials.

In this paper the test procedure and null distribution of the test statistic are studied in detail for the general three parameter model. Selected critical values of the test statistic are tabulated. The special case where $p=\beta$ is also metioned.

2. The Distribution.

Let $\{X_t,\ t=1,2,\cdots\}$ be a sequence of Bernoulli trials with Markov dependence such that

$$P[X_t=1\mid X_{t-1}=1]=\alpha, \qquad 0<\alpha<1,$$
$$P[X_t=1\mid X_{t-1}=0]=\beta, \qquad 0<\beta<1,$$
$$P[X_1=1]=p=1-q=1-P[X_1=0], \qquad 0<p<1,$$

where $t=2,3,\cdots\cdots$

Let the random variable N denote the total number of trials required to obtain a preassigned number $m\geq 2$ of successes. The joint distribution of $\{X_1,\cdots,X_N\}$ is then given, using the Markov dependence, by

$$\begin{aligned}
&P[X_N=x_N,\cdots,X_1=x_1]\\
&=P[X_N=x_N\mid X_{N-1}=x_{N-1}]\cdots P[X_2=x_2\mid X_1=x_1]\cdot P[X_1=x_1]\\
&=p^{x_1}q^{1-x_1}\prod_{t=2}^{N}[\alpha^{x_{t-1}x_t}(1-\alpha)^{x_{t-1}(1-x_t)}\beta^{(1-x_{t-1})x_t}(1-\beta)^{(1-x_{t-1})(1-x_t)}]\\
&=p^{x_1}q^{1-x_1}\alpha^{\Sigma_{t=2}^{N}x_{t-1}x_t}(1-\alpha)^{\Sigma x_{t-1}(1-x_t)}\beta^{\Sigma(1-x_{t-1})x_t}(1-\beta)^{\Sigma(1-x_{t-1})(1-x_t)}.
\end{aligned} \tag{1}$$

Let R be the random variable denoting the number of runs of successes. Then

$$R=\sum_{t=2}^{N}X_{t-1}(1-X_t)+1=m-\sum_{t=2}^{N}X_{t-1}X_t. \tag{2}$$

It can then be shown that (X_1,R,N) is a sufficient statistic for (p,α,β) with joint distribution

$$\begin{aligned}
p(x_1,r,n)&=P[X_1=x_1,\ R=r,\ N=n]\\
&=\binom{m-1}{r-1}\binom{n-m-1}{r-x_1-1}p^{x_1}q^{1-x_1}\alpha^{m-r}(1-\alpha)^{r-1}\beta^{r-x_1}(1-\beta)^{n-m-r+x_1}
\end{aligned} \tag{3}$$

where $x_1=0,1,\quad r=1,2,\cdots,m,\quad n=m,m+1,\cdots,\ \binom{-1}{-1}=1,\ \binom{k}{j}=0$ for $j>k$, and $\binom{k}{-1}=0$ for all $k\geq 0$.

The probability generating function $G_N(z)$ of N is found to be

$$\begin{aligned}
G_N(z)&=\sum_{n=m}^{\infty}z^n\sum_{m=0}^{1}\sum_{r=1}^{m}p(x_1,r,n)\\
&=[pz+q\beta z^2/(1-(1-\beta)z)]\cdot[\alpha z+(1-\alpha)\beta z^2/(1-(1-\beta)z)]^{m-1},
\end{aligned} \tag{4}$$

and the mean and variance of N are found to be

$$\begin{aligned}
E(N)&=m+[(1-\alpha)m+\alpha-p]/\beta,\\
\mathrm{Var}(N)&=[(1-(1-\alpha)\beta-\alpha^2)m+\alpha^2-\alpha\beta-p^2+p\beta]/\beta^2.
\end{aligned} \tag{5}$$

3. Test of Independence.

If $P[X_t=j\mid X_{t-1}=1]=P[X_t=j\mid X_{t-1}=0]$ for all $j=0,1$ and $t=2,3,\cdots$, that is, $\alpha=\beta$, then $\{X_t, t=2,3,\cdots\}$ are independent. Hence, a test of independence of the trials can be formulated in terms of a test of the null hypothesis H_0: $\alpha=\beta$ which in turn can be constructed based on the sufficient statistic (X_1,R,N). The statistic (X_1,N) forms, under H_0, a complete sufficient statistic. Hence, any unbiased test of H_0 is similar and therefore has Neyman Structure with respect to (X_1,N). Therefore, we can find a most powerful unbiased test of δ level by finding a most powerful conditional test of δ level on each of the surfaces $(X_1,N)=(x_1,n)$.

Bai and Rustagi (1974) have shown that, given $X_1=x_1$ and $N=n$, the conditional likelihood ratio is proportional to a function which is decreasing in R if $\alpha>\beta$, and increasing in R if $\alpha<\beta$. Hence, our test will reject H_0 if R is too small against alternative $\alpha>\beta$, and if R is too large against alternative $\alpha<\beta$.

Following Lehmann(1959) the test of H_0: $\alpha=\beta$ against these one-sided alternatives are characterized as follows:

Theorem. *Define the critical functions* ϕ_1 *and* ϕ_2 *by*

$$\phi_1(x_1,r,n)=\begin{cases}1 & r<c_1\\ \gamma_1 & r=c_1\\ 0 & r>c_1,\end{cases} \quad (6)$$

and

$$\phi_2(x_1,r,n)=\begin{cases}1 & r>c_2\\ \gamma_2 & r=c_2\\ 0 & r<c_2,\end{cases} \quad (7)$$

where $c_i=c_i(x_1,n)$ are nonnegative integer-valued and $\gamma_i=\gamma_i(x_1,n)$ take values in $(0,1)$, and they are determined by the condition

$$E_{H_0}[\phi_i(X_1,R,N)\,|\,x_1,n]=\delta. \quad (8)$$

Then ϕ_1 and ϕ_2 constitute uniformly most powerful unbiased level δ test for testing $H_0:\alpha=\beta$ against alternatives $\alpha>\beta$ and $\alpha<\beta$ respectively.

Notice that, for example, the critical function ϕ_1 specifies that H_0 is to be rejected or accepted according to whether the observed value of R is less than or greater than c_1 respectively. When the observed value of R is equal to c_1, H_0 is to be rejected with probability γ_1 and to be accepted with probability $1-\gamma_1$.

4. The Null Distribution.

The expectations (8) are taken with respect to the conditional distribution $P_H\ (r\,|\,x_1,n)$ of R given $X_1=x_1$ and $N=n$. Under H_0, (3) is reduced to

$$p_{H_0}(x_1,r,n)=\binom{m-1}{r-1}\binom{n-m-1}{r-x_1-1}p^{x_1}q^{1-x_1}\alpha^{m-x_1}(1-\alpha)^{n-m+x_1-1}$$

Summing it over r, we obtain

$$p_{H_0}(x_1,n)=\binom{n-2}{m-x_1-1}p^{x}q^{1-x_1}\alpha^{m-x_1}(1-\alpha)^{n-m+x_1-1}$$

Thus,

$$p_{H_0}(r\,|\,x_1,n)=\binom{m-1}{r-1}\binom{n-m-1}{r-x_1-1}\bigg/\binom{n-2}{m-x_1-1}. \quad (9)$$

The conditions (8) then become

$$\sum_{r=1}^{c_1-1}p_{H_0}(r\,|\,x_1,n)+\gamma_1 p_{H_0}(c_1\,|\,x_1,n)=\delta \quad (10)$$

and

$$\sum_{r=c_2}^{m}p_{H_0}(r\,|\,x_1,n)+\gamma_2 p_{H_0}(c_2\,|\,x_1,n)=\delta. \quad (11)$$

The distribution (9) takes several different forms depending on the values of x_1 and n;

$$p_{H_0}(r\,|\,x_1,n)=\begin{cases}\text{degenerate at } r=1 & \text{for } n=m,\\ \binom{m-1}{r-1}\binom{n-m-1}{r-2}\Big/\binom{n-2}{m-2} & \text{for } n>m,\ x_1=1,\\ \binom{m-1}{r-1}\binom{n-m-1}{r-1}\Big/\binom{n-2}{m-1} & \text{for } n>m,\ x_1=0,\end{cases} \quad (12)$$

and represents the conditional distribution of the number of runs of successes when the total number of trials is given and under the conditions that a) the trials are independent and have the same probability of a success except the initial trial and b) the trials are carried out by the inverse binomial type sampling scheme.

The factorial moments of the distributions are given by

$$\begin{aligned}E[(R-1)^{[k]}\,|\,x_1,n]&=E[(R-1)(R-2)\cdots(R-k)\,|\,x_1,n]\\ &=\sum_{r=k+1}^{m}(r-1)^{[k]}p_{H_0}(r\,|\,x_1,n)\\ &=\left[(m-1)^{[k]}\Big/\binom{n-2}{m-x_1-1}\right]\cdot\sum_{r=k+1}^{m}\binom{m-k-1}{r-k-1}\binom{n-m-1}{r-x_1-1}\\ &=(m-1)^{[k]}\cdot\binom{n-k-2}{m-x_1-1}\Big/\binom{n-2}{m-x_1-1}.\end{aligned} \quad (13)$$

In particular, for $n>m$, we obtain

$$E(R\,|\,x_1,n)=1+(m-1)(n-m+x_1-1)/(n-2) \quad (14)$$

and

$$\mathrm{Var}(R\,|\,x_1,n)=(m-1)(m-x_1-1)(n-m-1)(n-m+x_1-1)/(n-2)^2(n-3). \quad (15)$$

Selected values of c_1 and c_2 are tabulated in Table 1 and 2 for $\delta=0.05$ and $m=2$ to 10.

For large m, it can be shown that R is approximately (conditionally) normally distri-

TABLE 1 SELECTED VALUES OF c_1 FOR $\delta=.05$ AND $m=2$ TO 10

m	2		3		4		5		6		7		8		9		10	
n \ x_1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
3	1	2																
4	1	2	1	2														
5	1	2	1	2	1	2												
6	1	2	1	2	1	2	1	2										
7	1	2	1	2	1	2	1	2	1	2								
8	1	2	1	2	2	2	1	2	1	2	1	2						
9	1	2	2	2	2	2	2	2	2	2	1	2	1	2				
10	1	2	2	2	2	2	2	2	2	2	2	2	1	2	1	2		
11	1	2	2	2	2	2	2	3	2	3	2	3	2	2	1	2	1	2
12	1	2	2	2	2	2	2	3	2	3	2	3	2	3	2	2	1	2
13	1	2	2	2	2	2	2	3	2	3	2	3	2	3	2	3	2	2
14	1	2	2	2	2	3	2	3	3	3	3	3	3	3	2	3	2	3
15	1	2	2	2	2	3	2	3	3	3	3	3	3	3	3	3	2	3
16	1	2	2	2	2	3	3	3	3	3	3	3	3	3	3	3	3	3
17	1	2	2	2	2	3	3	3	3	3	3	4	3	4	3	4	3	4
18	1	2	2	2	2	3	3	3	3	3	3	4	3	4	3	4	3	4
19	1	2	2	2	2	3	3	3	3	3	3	4	3	4	4	4	4	4
20	1	2	2	2	2	3	3	3	3	4	3	4	4	4	4	4	4	4
25	2	2	2	2	3	3	3	3	4	4	4	4	4	5	4	5	5	5
30	2	2	2	2	3	3	3	4	4	4	4	4	5	5	5	5	5	6
40	2	2	2	2	3	3	4	4	4	4	5	5	5	5	6	6	6	6
50	2	2	2	3	3	3	4	4	4	5	5	5	5	6	6	6	7	7
100	2	2	3	3	3	3	4	4	5	5	6	6	6	6	7	7	8	8

TABLE 2 SELECTED VALUES OF c_2 FOR $\delta=.05$ AND $m=2$ TO 10

m	2		3		4		5		6		7		8		9		10	
n \ x_1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
3	1	2																
4	2	2	1	2														
5	2	2	2	3	1	2												
6	2	2	3	3	2	3	1	2										
7	2	2	3	3	3	4	2	3	1	2								
8	2	2	3	3	4	4	3	4	2	3	1	2						
9	2	2	3	3	4	4	4	4	3	4	2	3	1	2				
10	2	2	3	3	4	4	4	5	4	5	3	4	2	3	1	2		
11	2	2	3	3	4	4	4	5	4	5	4	5	3	4	2	3	1	2
12	2	2	3	3	4	4	5	5	5	5	5	5	4	5	3	4	2	3
13	2	2	3	3	4	4	5	5	5	5	5	6	5	5	4	5	3	4
14	2	2	3	3	4	4	5	5	5	6	5	6	5	6	5	6	4	5
15	2	2	3	3	4	4	5	5	5	6	6	6	6	6	5	6	5	6

16	2	2	3	3	4	4	5	5	6	6	6	6	6	7	6	7	6	6
17	2	2	3	3	4	4	5	5	6	6	6	6	6	7	6	7	6	7
18	2	2	3	3	4	4	5	5	6	6	6	7	7	7	7	7	7	7
19	2	2	3	3	4	4	5	5	6	6	6	7	7	7	7	7	7	7
20	2	2	3	3	4	4	5	5	6	6	6	7	7	7	7	8	7	8
25	2	2	3	3	4	4	5	5	6	6	7	7	7	8	8	8	8	9
30	2	2	3	3	4	4	5	5	6	6	7	7	8	8	8	9	9	9
40	2	2	3	3	4	4	5	5	6	6	7	7	8	8	9	9	10	10
50	2	2	3	3	4	4	5	5	6	6	7	7	8	8	9	9	10	10
100	2	2	3	3	4	4	5	5	6	6	7	7	8	8	9	9	10	10

buted with mean $1+m(1-\theta)$ and variance $m\theta(1-\theta)^2$ where $\theta=m/n$.

5. A Special Case.

Consider a special case of our model where $p=\beta$. Parameter estimation problems of this model have been studied by Narayana and Sathe (1961) and Bai and Rustagi (1974). When $p=\beta$, the distribution (3) is reduced to

$$p(r,n)=\binom{m-1}{r-1}\binom{n-m}{r-1}\alpha^{m-r}(1-\alpha)^{r-1}\beta^r(1-\beta)^{n-r+1}. \qquad (16)$$

Tests similar to (6) and (7) can also be constructed based on the conditional distribution $p_{H_0}(r|n)$ of R given $N=n$ under H_0; $\alpha=\beta$ where

$$p_{H_0}(r|n)=\binom{m-1}{r-1}\binom{n-m}{r-1}\Big/\binom{n-1}{m-1}. \qquad (17)$$

This represents the distribution of the number of successes when the total number of trials is given in a sequence of independent and identically distributed Bernoulli trials, and differs from the well-known distribution of runs of successes given by Mood (1940) in that the the trials are carried out under the inverse binomial type sampling scheme.

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