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On the ordinal spaces

by

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In this note we try to investigate the topological structures of the ordinal spaces. We consider the methods for topologizing the ordered sets, especially the well ordered sets that play an important role in the field of general topology.

For definitions and notations we refer toJ. Dugundji [4].

In the set of all real numbers the usual topology $\mathfrak{T}$ is introduced as follows. Let $\mathfrak{S}$ be the family of all sets of the forms $\{x\in R|x>a\}$ and $\{x\in R|x<b\}$ and let $\mathfrak{B}$ be the family of all finite intersections of members of $\mathfrak{S}$ The topology $\mathfrak{T}$ generated by the base $\mathfrak{B}$ is the usual topology for R. We try to generalize this concept to the set of ordinal numbers.

DEFINITION 1. Let Γ be any ordinal number and let X be the set $\{x|0\leqslant x\leqslant \Gamma$, x is a ordinal numbers$\}$ of ordinals. Let S be the family of all sets of the forms $\{x\in X|x>\alpha$ for some $\alpha\in X\}$ and $\{x\in X|x<\beta$ for some $\beta\in X\}$ and let $\mathfrak{B}$ be the family of all finite intersections of members of $\mathfrak{S}$ then $\mathfrak{B}$ is a base for some topology $\mathfrak{T}$ for X. We call this topological space (X, $\mathfrak{T}$) the ordinal spacc $[0,\Gamma]$.

Let $(\alpha_1,\beta_1)\cap\cdots\cap(\alpha_n,\beta_n)$ be a member of $\mathfrak{B}$ then there are the $\max\{\alpha_1,\cdots,\alpha_n\}=\alpha$ and mim $\{\beta_1,\cdots,\beta_n\}=\beta$ and we have $(\alpha,\beta)\subset(\alpha_1,\beta_1)\cap\cdots\cap(\alpha_n,\beta_n)$. Therefore the family of sets of the forms $(\alpha,\beta)=(\alpha,\beta+1]$ and $\{0\}$ is the basis of $\mathfrak{T}$ Note also that $[\alpha,\beta)$ is open iff $\alpha=0$ or if α has an immediate predecessor.

THEOREM 1. Let Ω be the first uncountable ordinal number, and let $[0,\Omega)$ be the subspace of the ordinal space $[0,\Omega]$. Then each continuous function $\varphi:[0,\Omega)\to R$ must be constant on a tail $[\beta,\Omega)$.

(Proof) We fisrt assert that for each $n\in Z^+=\{1,2,3,\cdots\}$ there is an $\alpha_n<\Omega$ such that for each $\xi>\alpha_n$, $|\varphi(\xi)-\varphi(\alpha_n)|\geqq\frac{1}{n}$. For, if this were not true, then there is $n_0\in Z^+$ such that for each $\alpha>\Omega$ we have $|\varphi(\xi)-\varphi(\alpha)|>\frac{1}{n}$ for some $\xi>\alpha$. Then we could use the induction to construct a sequence $\{\xi_i|i\in Z^+\}$ such that both $\xi_i<\xi_{i+1}$ and $|\varphi(\xi_{i+1})-\varphi(\xi_i)|\geqq\frac{1}{n_0}$ for each i: choosing $\xi_1=0$ and assuming $\xi_1,\cdots,\xi_k$ defined, let ξ_{k+1} be the first element in $[\xi_k,\Omega)$ satisfying the hypothesis. The seqnence $\{\xi_i\}$ thus defined would then have a least upper bound $\gamma<\Omega$ and the φ would not be continuous at γ. This follows from the fact that any basis neighborhood $(\eta,\gamma]$ contains some ξ_i and therefore all ξ_k with $k>i$, but image of this neighborhood by φ cannot be contained in the neighborhood $\left(\varphi(\gamma)-\frac{1}{3n_0},\ \varphi(\gamma)+\frac{1}{3n_0}\right)$ of $\varphi(\gamma)$. This means that φ is not continuous at γ. This is a contradiction. Therefore our assertion is established. Now let β be an upper bound of the $\{\alpha_n | n \in Z^+\}$, then $\beta < \Omega$ and φ is constant on $[\beta, \Omega)$. For if $\zeta \in [\beta, \Omega)$, then we have both $|\varphi(\varphi(\zeta) - \varphi(\alpha_n)| < \frac{1}{n}$ and $|\varphi(\beta) - \varphi(\alpha_n)| < \frac{1}{n}$ for every n, so $|\varphi(\zeta) - \varphi(\beta)| < \frac{2}{n}$ for all n. Hence $\varphi(\zeta) = \varphi(\beta)$.

THEOREM 2. The ordinal space $[0, \Gamma)$ is normal for each ordinal number Γ.

(Proof) Let A, B be disjoint closed sets in $[0, \Gamma)$. For each α in A, the set $\{\beta \in B | \beta < \alpha\}$ has the least upper bound b_α. Let $(\lambda, \mu]$ be any basic open set containing b_α, then $(\lambda, \mu] \cap \{\beta \in B | \beta < \alpha\} \neq \phi$ or $(\lambda, \mu] \cap B \neq \phi$. Therefore $b_\alpha \in \bar{B} = B$. Since $(b_\alpha, \alpha]$ is an open set containing no point of B, we get an open set $U = \bigcup \{(b_\alpha, \alpha] | \alpha \in A\}$ which contains A. Similarly we can get an open set $V = \bigcup \{(a_\beta, \beta] | \beta \in B\}$ which contains B. Now U and V are disjoint. For if $U \cap V \neq \phi$, then for some $\alpha \in A$ and $\beta \in B$ we have $(b_\alpha, \alpha] \cap (a_\beta, \beta] \neq \phi$ and assuming, say, that $\beta < \alpha$, this gives $\beta \in (b_\alpha, \alpha]$ which is impossible because of $(b_\alpha, \alpha] \cap B = \phi$. Hence $[0, \Gamma)$ is normal.

THEOREM 3. Let Ω be the first uncountable ordinal number, then the ordinal space $[0, \Omega]$ is paracompact.

(proof) Let $\{U_\alpha | \alpha \in I\}$ be any open covering. Since the family $\mathfrak{B} = \{(\lambda, \mu] | \lambda, \mu \in [0, \Omega], \lambda < \mu\}$ form a basis, define $\varphi: [0, \Omega] \to [0, \Omega]$ by associating with each $\beta \neq 0$ a $\varphi(\beta) < \beta$ such that $(\varphi(\beta), \beta] \subset U_\alpha$ for some $\alpha \in I$, and setting $\varphi(0) = 0$. By induction, Let us construct a sequence $\beta_0 = \Omega$, $\beta_1 = \varphi(\beta_0), \cdots, \beta_n = (\beta_{n-1}), \cdots$ then $\beta_0 > \beta_1 > \cdots$. Since every descending sequence of ordinal numbers is finite, this terminates with some β_n. Hence $\beta_n = 0$, and $(0, \Omega] \subset \bigcup (\beta_i, \beta_{i-1}]$. Choosing a U_{α_i} containing $(\beta_i, \beta_{i-1}]$, $i = 1, 2, \cdots, n$, and some $U_{\alpha_0} \ni 0$, we have a finite subcovering $\{U_{\alpha_0}, U_{\alpha_1}, \cdots, U_{\alpha_n}\}$ of $\{U_\alpha | \alpha \in I\}$, which is consequently an open neighbourhood-finite refinement. Therefore $[0, \Omega]$ is paracompact.

COROLLARY. The ordinal space $[0, \Omega]$ is compact.

It can be easily seen that the ordinal space $[0, \Omega)$ is not paracompact. Let $\mathfrak{A} = \{[0, \alpha) | 0 < \alpha < \Omega\}$ be the open covering of $[0, \Omega)$, and let $\{U_\alpha | \alpha \in I\}$ be an open neighbourhood-finite refinement of $\mathfrak{A}$. If we define $\varphi: [0, \Omega) \to [0, \Omega)$ as in the proof of the theorem 3, there must be some β_0 such that for each λ there is $\beta > \lambda$ with $\varphi(\beta) \leqq \beta_0$. And it follows that $\beta_0 + 1$ is contained in infinitely many U_α. This contradicts the neighborhood-finiteness of $\{U_\alpha | \alpha \in I\}$.

THEOREM 4. The ordinal space $[0, \Omega]$ is not metrizable.

(Proof) We will show that $[0, \Omega]$ is not perfectly normal since every metric space is perfectly normal. Since the set $\{x | x < \Omega\}$ is open. its complement $\{\Omega\}$ is closed. Let $\{\Omega\}$ be a G_δ set, then there is a countable collection $\{G_i | i \in Z^+\}$ of open sets such that $\bigcap_{i=1}^{\infty} G_i = \{\Omega\}$. For each i there is a basic open set $(\alpha, \Omega] \subset G_i$. Being countable, the collection $\{\alpha_i | i \in Z^+\}$ has an upper bound $\alpha < \Omega$, so $\bigcap_{i=1}^{\infty} \{G_i\} \supset (\beta, \Omega] \neq \{\Omega\}$. This is the contradiction. Therefore $\{\Omega\}$ is not G_δ set, that is, $[0, \Omega]$ is not perfectly normal.

References

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