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SEMI-CLOSED SETS AND SEMI-CONTINUITY IN TOPOLOGICAL SPACES

by

Chang Joo-Sup

Yonsei University, Seoul, Korea.

1. Semi-Closed sets.

Let $T=(X, \mathfrak{T})$ denote a topological space. Instead of using the notion of a semi-open set which has been developed by Norman Levine since 1963, we will use semi-closed set by duality.

DEFINITION 1. A set A in T will be termed semi-closed (written S.C.) if and only if there exists an closed set F such that $\text{Int}(F) \subset A \subset F$ where Int denotes the interior operator in T.

DEFINITION 2. S.C. (T) will denote the class of all semi-closed sets in T.

THEOREM 1. $A \in$ S.C. (T) if and only if $\text{Int}(\text{Cl}(A)) \subset A$, Cl denoting the closure operator.

Proof. ($\Longrightarrow$) : Let $A \in$ S.C. (T). By Definition, there exists a closed set F such that $\text{Int}(F) \subset A \subset F$. Since $\text{Cl}(A) \subset F$, so $\text{Int}(\text{Cl}(A)) \subset \text{Int}(F)$. Hence $\text{Int}(\text{Cl}(A)) \subset \text{Int}(F) \subset A$.

($\Longleftarrow$) : Let $\text{Int}(\text{Cl}(A)) \subset A$. Then $\text{Int}(\text{Cl}(A)) \subset A \subset F$ for $F=\text{Cl}(A)$. Hence $A \in$ S.C. (T).

THEOREM 2. If $A_\alpha \in$ S.C. (T) for each $\alpha \in \Delta$, then $\bigcup_{\alpha \in \Delta} A_\alpha \in$ S.C. (T).

Proof. By assumption, there exists a closed set F_α such that $\text{Int}(F_\alpha) \subset A_\alpha \subset F_\alpha$ for each $\alpha \in \Delta$. Then $\text{Int}(\bigcup_{\alpha \in \Delta} F_\alpha) \subset \bigcup_{\alpha \in \Delta} \text{Int}(F_\alpha) \cup \bigcup_{\alpha \in \Delta} A_\alpha \subset \bigcup_{\alpha \in \Delta} F_\alpha$. Hence $\bigcup_{\alpha \in \Delta} A_\alpha \in$ S.C. (T) when $F=\bigcup_{\alpha \in \Delta} F_\alpha$.

THEOREM 3. Let $T=(X, \mathfrak{T})$ be a topological space and $T^*=(X^*, \mathfrak{T}^*)$ be a subspace of T such that $A \subset T^* \subset T$. If $A \in$ S.C. (T). Then $A \in$ S.C. (T^*).

Proof. Since $A \in$ S.C. (T), there exists a closed set F in T such that $\text{Int}_X(F) \subset A \subset F$ where Int_X denotes the interior operator in X. Now $F \subset X^*$ and thus $F=F \cap X^* \supset A \cap X^* = A \supset \text{Int}_X(F) \cap X^* = \text{Int}_{X^*}(F)$. So $\text{Int}_{X^*}(F) \subset A \subset F$. Hence $A \in$ S.C. (T^*).

2. Semi-Continuity.

DEFINITION 3. A set $N_p \subset X$ is a semi-nbd of a point $p \in X$ if and only if $p \in A \subset N_p$ for some $A \in$ S.C. (T). Henceforth we will use nbd to abbreviate neighborhood.

THEOREM 4. If $K \subset X$, then $K \in$ S.C. (T) if and only if K is a seme-nbd of each $p \in K$.

Proof. ($\Longrightarrow$) : Since $K \in$ S.C. (T), so there exists F such that $\text{Int}(F) \subset K \subset F$. Then for $A=\text{Int}(F)$ we have $p \in A = \text{Int}(F) \subset K$.

($\Longleftarrow$) : If K is a semi-nbd of each $p \in K$, then there exists $K_p \in$ S.C. (T) such that $p \in K_p \subset K$ for each $p \in K$. Hence, $\bigcup_{p \in K} K_p = K \in$ S.C. (T) by theorem 2.

DEFINITION 4. A function $f: (X, \mathfrak{T}) \rightarrow (Y, \mathfrak{T}^*)$ is semi-continuous at $p \in X$ if and only if $f^{-1}(N_{f(p)})$

is a semi-nbd of p for each nbd $N_{f(p)}$ of f(p). f is semi-continuous at each point of X.

THEOREM 5. Let $T=(X, \mathfrak{T})$ and $T^*=(X^*, \mathfrak{T}^*)$ are topological spaces. A function $f:T\to T^*$ is semi-continuous on X if and only if for $f(p)\in F^*$, there exists an $A\in S.C.(T)$ such that $p\in A$ and $f(A) \subset F^*$.

Proof. ($\Longrightarrow$) : Let $f(p)\in F^*$, then $p\in f^{-1}(F^*)\subset S.C.(T)$ since $f:T\to T^*$ is semi-continuous. Let $A= f^{-1}(F^*)$, then $p\in A$ and $f(A)=f(f^{-1}(F^*))\subset F^*$.

($\Longleftarrow$) : Let F^* be closed in T^* and let $p\in f^{-1}(F^*)$. Then $f(p)\in F^*$ and, by assumption, there exists an $A_p\in S.C.(T)$ such that $p\in A_p$ and $f(A_p)\subset F^*$. Thus $p\in A_p\subset f^{-1}(F^*)=\bigcup_{p\in f^{-1}(F^*)} A_p$. Since $A_p \in S.C.(T)$, so $f^{-1}(F^*)\in S.C.(T)$ by theorem 2.

REMARK. In the Theorem 5, semi-continuity can be characterized by the concept of semi-nbd instead of the concept of semi-closedness.

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