FOX H-FUNCTION AND THE TEMPERATURE IN A SLAB WITH FACES AT TEMPERATURE ZERO

BY G. K. DHAWAN AND D. D. PALIWAL

1. Introduction.

In recent years a number of authors have used Meijer G-function, H-function of Fox in heat conduction problems of bar, cylinder etc [1, 5].

The H-function introduced by Fox [4, p. 408] will be represented and defined as follows:

\[
H_{p, q}^{m,n} \left[ z \left\{ (a_p, e_p) \right\} \right] = \frac{1}{2\pi i} \int_L \prod_{j=m+1}^{p} \frac{\Gamma'(b_j - f_js)}{\Gamma'(1 - b_j + f_js)} \prod_{h=1}^{q} \frac{\Gamma'(1 - a_j + e_js)}{\Gamma'(a_j - e_js)} \, ds
\]

Where \( z \) is not equal to zero and an empty product is interpreted as unity; \( p, q, m \) and \( n \) are integers satisfying \( 0 \leq m \leq q, 0 \leq n \leq p; \) \( e_j (j=1, 2, \cdots p), f_k (h=1, \cdots q) \) are positive numbers and \( a_j (j=1, \cdots p), b_k (h=1, \cdots q) \) are complex numbers. \( L \) is the path of integration separating the increasing and decreasing sequences of the poles of the integrand. The parameters \( \{(a_p, e_p)\} \) represent \( (a_1, e_1) \cdots (a_p, e_p) \). These assumptions for the H-function will be adhered to throughout this paper.

As an example of the application of H-function of Fox in applied mathematics, we shall consider the problem of determining the temperature in a slab of homogeneous material bounded by the planes \( x=0 \) and \( x=\pi \) having an initial temperature \( u=f(x) \), varying only with the distances from the faces and with its two faces kept at zero temperature.

The formula for the temperature \( u \) at any instant and at all points of the slab is to be determined. In the problem it is clear that the temperature function is of the variables \( x \) and \( t \) only. Hence at each interior point this function \( u(x, t) \) must satisfy the heat equation for one dimensional form.

\[
\frac{\partial u}{\partial t} = K \frac{\partial^2 u}{\partial x^2} (0<x<\pi, \ t>0)
\]

In addition, it must satisfy the conditions.

\[
u(x, +0, t) = 0
\]

\[
u(x, -0, t) = 0 \quad (t>0)
\]

\[
u(x, +0) = f(x) \quad (0<x<\pi)
\]

The boundary value problem (1.2) – (1.4) is also the problem of temperatures in a right prism or cylinder whose length is \( \pi \) (taken so for conveniences in the computation), provided its later surface is insulated. Its ends \( x=0 \) and \( x=\pi \) are held at temper-
nature zero and its initial temperature is \( f(x) \).

In section 2 of this paper, we have evaluated an integral involving Fox \( H \)-function which is required in the proof of subsequent section.

Here \( a_p \) denotes \( a_1, \ldots, a_p \), \( \partial \) is positive integer and the symbol \( \mathcal{D}(\partial, a) \) represents the set of parameter \( \frac{a}{\partial}, \frac{a+1}{\partial}, \ldots, \frac{a+\partial-1}{\partial} \).

In this paper we shall consider.

\[
(1.5) \quad f(x) = (\sin \frac{x}{2})^{2n-\beta-1} (\cos \frac{x}{2})^{\beta-1} H_{p, q}^{m, n} \left[ z \left( \tan \frac{x}{2} \right)^{2z} \right] \frac{\{(a_p, e_p)\}}{\{(b_q, f_q)\}}
\]

2. The integral.

\[
(2.1) \quad \int_0^\pi \sin \theta \left( \sin \frac{\theta}{2} \right)^{2n-\beta-1} (\cos \frac{\theta}{2})^{\beta-1} H_{p, q}^{m, n} \left[ z \left( \tan \frac{\theta}{2} \right)^{2z} \right] \frac{\{(a_p, e_p)\}}{\{(b_q, f_q)\}} d\theta
\]

\[
= \frac{\Gamma(2n)}{(2\pi)^{2n-\beta}} H_{\frac{2n}{2}, \frac{\beta}{2}+rac{\beta}{2}} \left[ z \left( \tan \frac{\theta}{2} \right)^{2z} \right] \frac{\mathcal{D}(\partial, 1+\beta-2n)}{\mathcal{D}(2\partial, \beta)} \frac{(a_p, e_p), \mathcal{D}(\partial, 2+\beta-2n)}{(b_q, f_q)}
\]

where

\[
\phi = \sum_{j=1}^{n} e_j - \frac{\xi}{2n-\sum_{j=1}^{n} e_j + \frac{\xi}{2n}}, \quad |\arg z| < \phi \cdot \frac{\pi}{2}
\]

and \( 2n > \text{Re}(\beta) > 0, n = 1, 2, 3, \ldots \)

**Proof.** To establish (2.1) express the \( H \)-function as a Mellin Barnes type integral [4, p. 408] and interchange the order of integration, which is justified due to the absolute convergence of the integrals involved in the process, we have

\[
\frac{1}{2\pi i} \int_L \left[ \prod_{j=1}^{m} \Gamma(b_j - f_j) \right] \left[ \prod_{j=1}^{n} \Gamma(1 - a_j + e_j) \right] \frac{d\zeta}{\zeta}
\]

\[
\int_0^\pi \sin \theta \left( \sin \frac{\theta}{2} \right)^{2n+2\beta-\beta-1} (\cos \frac{\theta}{2})^{\beta-2\beta-1} d\theta
\]

Now evaluating the inner integral with the help of the modified form of the formula [6] namely.

\[
\int_0^\pi \sin \theta \left( \sin \frac{\theta}{2} \right)^{2n-\beta-1} (\cos \frac{\theta}{2})^{\beta-1} d\theta = \frac{2^{2n-\beta-1} \Gamma(\pi) \Gamma(\frac{2n-\beta+1}{2}) \Gamma(\beta)}{\Gamma(1+\frac{\beta}{2}-n) \Gamma(2n)} \quad (2n > \text{Re}(\beta) > 0)
\]

and using the multiplication formula for the gamma function [3, p. 4(11)] we get
Fox $H$-function and the temperature in a slab with faces at temperature zero

\[
\frac{(2\beta)^{2n-1}}{(2\pi)^{\beta-1}I(2n)} \times \int L \prod_{j=1}^{\infty} \Gamma(b_j - f_j, f_j) \prod_{i=1}^{\infty} \Gamma(1 - a_j + ej, a_j - ej) \prod_{i=1}^{\infty} \Gamma(2 + \beta - 2n - i, \frac{2}{\beta} - i - s) \prod_{i=1}^{\infty} \Gamma(2n - \beta + 1, i - s) ds
\]

- on applying [4, p. 408] the value of the integral (2.1) is obtained.

3. The solution of problem.

The solution of the problem is

\[
(3.1) \quad u(x, t) = \frac{4}{(2\pi)^{\frac{1}{2}}} \sum_{n=1}^{\infty} (2\beta)^{n-1} e^{-\frac{x^2}{2}} \sin x H_{\beta - 2n + 1} \left( x, \frac{x}{2} \right) \sin x \sin x dx
\]

where

\[
\phi = \sum_{j=1}^{\infty} e_j - \sum_{j=1}^{\infty} f_j > 0, \quad |\arg x| < \phi \cdot \frac{\pi}{2}
\]

and

\[
\text{Re} (\beta) > 0.
\]

**Proof.** The solution of the problem can be written as [2, p. 139(6)]

\[
(3.2) \quad u(x, t) = \sum_{i=1}^{\infty} A_i \exp (-s^2 kt) \sin x
\]

If \( t = 0 \), then by virtue of (1.5), we have

\[
(3.3) \quad (\sin \frac{x}{2})^{2n-\beta-1} (\cos \frac{x}{2})^{\beta-1} H_{\beta - 2n + 1} \left( x, \frac{x}{2} \right) \sin x (\tan \frac{x}{2})^{2n} \left( \{a_{\beta, \beta}\} \right) = \sum_{i=1}^{\infty} A_i \sin x
\]

Multiplying both sides of (3.3) by \( \sin nx \) and integrating with respect to \( x \) from 0 to \( \pi \) we get

\[
(3.4) \quad \int_{0}^{\pi} \sin nx \sin \left( \sin \frac{x}{2} \right)^{2n-\beta-1} (\cos \frac{x}{2})^{\beta-1} H_{\beta - 2n + 1} \left( x, \frac{x}{2} \right) \sin x (\tan \frac{x}{2})^{2n} \left( \{a_{\beta, \beta}\} \right) dx
\]

\[
= \sum_{i=1}^{\infty} A_i \int_{0}^{\pi} \sin x \sin nx dx
\]

Now using (2.1) and the orthogonal property of the sine function we have

\[
(3.5) \quad A_i = \frac{(2\beta)^{2n-1}}{(2\pi)^{\beta-1}I(2n)} H_{\beta - 2n + 1} \left( x, \frac{x}{2} \right) \sin x \sin \left( \sin \frac{x}{2} \right)^{2n-\beta-1} (\cos \frac{x}{2})^{\beta-1} H_{\beta - 2n + 1} \left( x, \frac{x}{2} \right) \sin x (\tan \frac{x}{2})^{2n} \left( \{a_{\beta, \beta}\} \right) \sin x (\tan \frac{x}{2})^{2n} \left( \{a_{\beta, \beta}\} \right)
\]
with the help of (3.2) and (3.5) the solution (3.1) is obtained.

4. Conclusion.

On specializing the parameter the $H$-function may be converted into $G$-function, Bessel function, Legendre function and other higher transcendental functions [3, pp216–222]. Therefore the function $f(x)$ given in (1.5) is of general character and hence may encompass several cases of interest.

References


M. A. C. T. and S. V. Government Polytechnic, Bhopal, India.