A NOTE ON $L^2$-CONTINUITY OF PSEUDODIFFERENTIAL OPERATORS

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1. Introduction. A pseudodifferential operator $A \in L^p_{\rho, \delta} : S \rightarrow S'$ is defined by

\begin{equation}
Au(x) = (2\pi)^{-n} \int \delta(x-x_0) \cdot \xi a(x, x_0, \xi) u(x_0) dx_0 d\xi
\end{equation}

where the symbol $a(x, x_0, \xi)$ belongs to the space $S^p_{\rho, \delta}$ (see [5], [6], [7]), namely

\begin{equation}
|\partial_{x_j}^\alpha \partial_{\xi_j}^\beta a(x, x_0, \xi)| \leq C_{\alpha\beta}(1 + |\xi|^m) a_1^\alpha j_\alpha + a_2^\beta j_\beta, \quad j=1, 2
\end{equation}

In particular if $a(x, x_0, \xi)=P(\xi)$, a polynomial in $\xi$, then $A$ coincides with the differential operator $P(D), D=(-i\partial_{x_1}, \ldots, -i\partial_{x_n})$.

Hörmander ([6]) showed that (1.1) is $L^2$-continuous if $a(x, x_0, \xi)$ has a compact support in $(x_1, x_2)\,-$variables, satisfies (1.2) for all multiindices $\alpha, \beta$, and $m<\rho-(1/2)(\delta_1+\delta_2)$ where $0<\rho<\delta_j<1$. He also proved that (1.1) fails to be $L^2$-continuous if $m>n\rho-(1/2)\delta_j$, $j=1, 2$, and removed the assumption of compact supportness in $\xi\,-$variable. They also settled down the critical case of $m=n\rho-(1/2)\delta_j$. Cordes ([4]) proved similar results with $0 \leq |\alpha|, |\beta| \leq [n/2]+1$, for a symbol $a(x, \xi)$ of 2n-variables using Banach algebra techniques. Childs ([3]) improved Cordes' results by assuming instead multiple Hölder continuity of order greater than 1/2. Kato ([8]) also proved similar theorems to Cordes employing Banach algebra techniques and partition of unity. It is known that $L^2$-continuity of (1.1) fails for the spaces $S^0_{\rho, \delta}$, $0<\rho<1$ (Kumano-Go ([8])) and $S^1_{\rho, \delta}$, (Chin-Hung Ching, Ph. D. Thesis, New York University, 1971).

In this paper we further improve Calderón and Vaillancourt's result on the order of differentiabilities with respect to the $x_j,-$variables ($j=1, 2$), and drop the assumption of compact supportness in $\xi,-$variable. In the end we list two open problems related to this paper.

2. Main Theorem. Theorem. Let $a(x, x_0, \xi)$ be compactly supported in $\xi,-$variable, and satisfy (1.2) for $0 \leq |\alpha| \leq 2m_j, 0 \leq |\beta| \leq 2[n/2]+2, m_j (j=1, 2)$ is the smallest integer $\geq (5n)/(4(1-\delta_j)), j=1, 2$, and removed the assumption of compact supportness in $(x_1, x_2)\,-$variables while assuming compact supportness in $\xi\,-$variable. They also settled down the critical case of $m=n\rho-(1/2)\delta_j$. Cordes ([4]) proved similar results with $0 \leq |\alpha|, |\beta| \leq [n/2]+1$, for a symbol $a(x, \xi)$ of 2n-variables using Banach algebra techniques. Childs ([3]) improved Cordes' results by assuming instead multiple Hölder continuity of order greater than 1/2. Kato ([8]) also proved similar theorems to Cordes employing Banach algebra techniques and partition of unity. It is known that $L^2$-continuity of (1.1) fails for the spaces $S^0_{\rho, \delta}$, $0<\rho<1$ (Kumano-Go ([8])) and $S^1_{\rho, \delta}$, (Chin-Hung Ching, Ph. D. Thesis, New York University, 1971).

In this paper we further improve Calderón and Vaillancourt's result on the order of differentiabilities with respect to the $x_j,-$variables ($j=1, 2$), and drop the assumption of compact supportness in $\xi,-$variable. In the end we list two open problems related to this paper.

Proof. We essentially follow Calderón and Vaillancourt's proof ([2]). It suffices to prove the theorem for $m=n\rho-(1/2)(\delta_1+\delta_2)$ because the right hand side of (1.2) is decreasing as $m$ decreases. Let $N=[n/2]+1$. Notice first the identity

\begin{equation}
(1+(1+|\xi|)^{2N}\rho|x_1-x_2|^{2N})^{-1}(1+(1+|\xi|)^{2N}\rho(-\Delta)^{N})e^{i(\xi_1-x_2)\cdot \xi} = e^{i(x_1-x_2)\cdot \xi}
\end{equation}

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where $\Delta = \sum \frac{\partial^2}{\partial x_i^2}$ is the Laplacian.

Integration by parts of (1.1) yields

$$Au(x_1) = (2\pi)^{-n} \int e^{i(x_1-x_2).\xi} \ b(x_1, x_2, \xi) \ u(x_2) \ dx_2 d\xi$$

where $b(x_1, x_2, \xi) = [1 + (-\Delta)^{N/2} (1 + |\xi|) |x_1 - x_2|^{-N}]^{-1}.$

According to [2: pp. 1185-1186], $b(x_1, x_2, \xi)$ is majorized by

$$|\partial_\xi^\alpha b(x_1, x_2, \xi)| \leq C \|a\|_1 (1 + |\xi|)^{N + \alpha} G((1 + |\xi|)^{\alpha} |x_1 - x_2|)$$

for $0 \leq |\alpha| \leq 2m$, $m_j$ is to be determined later, $\|a\|_1 = \inf C_{ab}$ in the right hand side of (1.2), and $G$ is an integrable function such that

$$\int G(x) dx \leq 1$$

The following lemma was verified by Calderón and Vaillancourt ([1: pp. 374-378], [2: p. 1185]):

**Lemma (Calderón and Vaillancourt).** Let $A$ be a bounded operator on a Hilbert space and let $A(x)$ be a weakly-measurable, uniformly bounded operator valued function on a measure space $X$ with measure $dx$. If

$$\|A^*(x) A(x_2)\| \leq h_2(x_1, x_2)$$

$$\|A(x_1) A^*(x_2)\| \leq h_2(x_1, x_2)$$

and if

$$\int h_1(x_1, x) h_2(x, x_2) dx$$

is the kernel of a bounded operator on $L^2(X)$ with norm $M^2$, then

$$\|\int_B A(x) dx\| \leq M$$

where $B$ is any Borel subset of finite measure of $X$.

To apply the lemma, first observe that the kernel of $A$ is given by

$$A(\xi) u = (2\pi)^{-n} \int e^{i(x_1-x_2).\xi} b(x_1, x_2, \xi) u(x_2) dx_2$$

It follows that the kernel of $A^*(\xi_1) A(\xi_2)$ is given by

$$(2\pi)^{-n} \int e^{-i\xi_2.\xi_1} e^{i(x_1-x_2).\xi_2} b(y, x_1, \xi_1) b(y, x_2, \xi_2) dy$$

Noting the identity $|\xi_2 - \xi_1|^{-2m_1} (-\Delta)^{m_1} e^{i(x_2-x_1).\xi_1} = e^{i(x_2-x_1).\xi_1}$, integration by parts yields

$$(2\pi)^{-n} \int e^{-i\xi_2.\xi_1} e^{i(x_1-x_2).\xi_2} |\xi_2 - \xi_1|^{-2m_1} (-\Delta)^{m_1} b(y, x_1, \xi) b(y, x_2, \xi_2) dy$$

In view of (2.1) and Leibniz rule, (2.2) is majorized by
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\[
C\|a\|^2(1 + |\xi_1|)^N(1 + |\xi_2|)^N(1 + |\xi_1| + |\xi_2|)^{2m_1}d_1|\xi_1 - \xi_2|^{-2m_1}.
\]

\[
\int G((1 + |\xi_1|)\eta(y-x_1))G((1 + |\xi_2|)\eta(y-x_2))dy
\]

Since $\int G(y)dy \leq 1$, change of variables and Schwarz inequality reveal

\[
\|A^*(\xi_1) A(\xi_2)\| \leq h_2^2(\xi_1, \xi_2)
\]

where

\[
h_2^2(\xi_1, \xi_2) = C\|a\|^2(1 + |\xi_1|)^N(1 + |\xi_2|)^N(1 + |\xi_1| + |\xi_2|)^{2m_1}d_1|\xi_1 - \xi_2|^{-2m_1}.
\]

Likewise,

\[
\|A(\xi_1) A^*(\xi_2)\| \leq h_2^2(\xi_1, \xi_2)
\]

where $h_2$ is obtained from $h_1$ by replacing $d_1$ and $m_1$ by $d_2$ and $m_2$ respectively. Set

\[
h_j^{(1)}(\xi_1, \xi_2) = (1 + |\xi_1|)^{-\frac{N}{2}(d_1+d_2)}(1 + |\xi_2|)^{-\frac{N}{2}(d_1+d_2)} - m_j(1 - d_j) \chi_1(\xi_1, \xi_2)
\]

\[
h_j^{(2)}(\xi_1, \xi_1) = (1 + |\xi_1|)^{-\frac{N}{2}(d_1+d_2)}[1 + \frac{|\xi_1 - \xi_2|^{2m_j}}{(1 + |\xi_1| + |\xi_2|)^{2m_jd_j}}]^{-1/2} \chi_2(\xi_1, \xi_2)
\]

\[
h_j^{(3)}(\xi_1, \xi_2) = (1 + |\xi_1|)^{-\frac{N}{2}(d_1+d_2)} - m_j(1 - d_j) (1 + |\xi_2|)^{-\frac{N}{2}(d_1+d_2)} \chi_3(\xi_1, \xi_2)
\]

where $\chi_1$, $\chi_2$ and $\chi_3$ are the characteristic functions of the sets

\[
\{(\xi_1, \xi_2) : |\xi_1| \leq \frac{1}{2} |\xi_2| \},
\]

\[
\{(\xi_1, \xi_2) : \frac{1}{2} |\xi_2| \leq |\xi_1| \leq 2 |\xi_2| \},
\]

\[
\{(\xi_1, \xi_2) : 2 |\xi_2| \leq |\xi_1| \}
\]

respectively. It is readily seen that

\[
h_j \leq C\|a\|(h_j^{(1)} + h_j^{(2)} + h_j^{(3)})
\]

A simple computation shows for $j=1, 2, 3$

\[
\int h_1^{(j)}(\xi_1, \xi_2) d\xi_1 d\xi_2 \leq C(1 + |\xi_2|)^{-\frac{N}{2}(d_1+d_2)} - m_j(1 - d_j) + n
\]

\[
\int h_2^{(j)}(\xi_1, \xi_2) (1 + |\xi_1|)^{-\frac{N}{2}(d_1+d_2)} - m_j(1 - d_j) + n d\xi_1
\]

\[
\leq C(1 + |\xi_2|)^{-\frac{N}{2}(d_1+d_2)} - m_j(1 - d_j) + m_j(1 - d_j) + 2n \leq C
\]

Similar inequalities are obtained when we integrate against the second variable. In this case we have to replace $m_1$ and $d_1$ by $m_2$ and $d_2$ respectively. Notice that the integrals (2.3) and (2.4) converge if and only if $m_j \geq -n{1 - 1/(4d_1 + d_2)}$ \quad (j=1, 2), which subsequently imply $n(\delta_1 + \delta_2) + m_1(1 - \delta_1) + m_2(1 - \delta_2) - 2n > 0$, and hence the right hand side of (2.4) is bounded by a constant. Consequently we have from (2.3) and (2.4)

\[
\int h_1(\xi_1, \xi) h_2(\xi, \xi_2) d\xi_1 d\xi \leq C\|a\|^2
\]

\[
\int h_1(\xi_1, \xi) h_2(\xi, \xi_2) d\xi_2 \leq C\|a\|^2
\]
Observe that the cross product terms vanish in the above integrals. It follows from the lemma that \( A = \int A(\xi) d\xi \) is \( L^2 \)-continuous whose norm is bounded by \( C\|a\| \). This completes the proof.

**Remark.** Our estimate is better than that of Calderón and Vaillancourt's by
\[
\frac{n(1+\delta_1+\delta_2)}{4(1-\delta_1)} \quad (j=1,2).
\]

**Corollary.** The theorem still holds without the assumption that \( a(x_1, x_2, \xi) \) has a compact support in \( \xi \)-variable.

**Proof.** Following [6] and [7], we do a partition of unity as follows: Let \( \epsilon \in C_0^\infty (R^l) \) be a nonnegative radial function such that \( \epsilon(r) > 0 \) for \( r < 1/2 \), \( \epsilon(r) = 0 \) for \( r > 3/4 \). Let \( f(\xi) = \epsilon(|\xi|) \) and define \( \varphi_k(\xi) = f(\xi-k)[\sum_0^\infty f(\xi-k)]^{-1} \). Then \( \{\varphi_k\} \) is a partition of unity.

Furthermore,
\[
|\partial^\beta \varphi_k(\xi)| \leq C_\beta
\]

Let \( \Psi_k(\xi) = \varphi_k(\xi |\xi|^{-\rho}) \). Then \( \{\Psi_k\} \) is also a partition of unity and (2.5) implies
\[
|\partial^\beta \Psi_k(\xi)| \leq C_\beta (1+|\xi|)^{-\rho^1 \beta_1}
\]

Choose a sequence \( \{\xi_k\} \), \( \xi_k \in \text{supp } \Psi_k \) such that
\[
|\xi_k| \geq C_1 k
\]
\[
|\xi - \xi_k| \leq C_1 |\xi_k|^\rho, \quad \xi \in \text{supp } \Psi_k.
\]

Here \( C_1 \) and \( C_2 \) are independent of \( k \). Now set \( a_k(x_1, x_2, \xi) = a(x_1, x_2, \xi) \Psi_k(\xi) \). Then each \( a_k(x_1, x_2, \xi) \) has a compact support in \( \xi \). Moreover, (1.2), (2.6) and Leibniz rule show that \( a_k(x_1, x_2, \xi) \) belongs to the space \( S_{\rho}^m \). Let \( A_k \) be the corresponding operator associated with \( a_k(x_1, x_2, \xi) \), namely
\[
A_k u(x_1) = (2\pi)^{-n} \int e^{i(x_1-x_2) \cdot \xi} a_k(x_1, x_2, \xi) u(x_2) dx_2 d\xi
\]

The kernel of \( A_k \) is given by
\[
A_k(x_1, x_2) = (2\pi)^{-n} \int e^{i(x_1-x_2) \cdot \xi} a_k(x_1, x_2, \xi) d\xi
\]

Integration by parts \( s \) times we have in view of (1.2), (2.6) and (2.8)
\[
|A_k(x_1, x_2)| \leq C_1 (1+|x_1-x_2|)^{-s}(1+|\xi_k|)^{n-ps+ps}
\]

By taking \( s = n + M \) where \( M \) is a positive integer such that \( \rho M > 1 \) for \( 0 < \rho < 1 \), we obtain
\[
A_k(x_1, x_2) \ dx_2 \leq C(1+|\xi_k|)^{n-pM}
\]
\[
A_k(x_1, x_2) \ dx_2 \leq C(1+|\xi_k|)^{n-pM}
\]
It follows that

\[(2.9) \qquad \| A \| \leq C (1 + | \xi_k |)^{-m} \rho M \]

Recalling \( m = n \{ \rho - \frac{1}{2} (\partial_1 + \partial_2) \} \leq 0, \) and \( | \xi_k |^{-1} \leq C k^{-1} \) from (2.7), we finally have from (2.9)

\[ \| A \| \leq \sum_{k} \| A_k \| < \infty, \]

thereby completing the proof.

In [10], Fefferman introduced wave packet transform whose kernel is in fact a symbol of \( 3n \)-variables in \( S^{1/2}_{1/2} \).

3. Some open problems. In [8] Kato verified \( L^2 \)-continuity of pseudodifferential operators associated with symbols \( a(x, \xi) \) in \( S^0 \) of \( 2n \)-variables using Banach algebra techniques and partition of unity. This argument heavily depends on the fact that the convolution \( a * g \) defined by

\[ (a * g) (x, D) = \int a (x, \xi) \hat{e}^x \cdot x e^{-ix \cdot D} g (x, D) e^{ix \cdot D} e^{-ix \cdot x} dx d\xi \]

exists as a strong integral for \( a(x, \xi) \in L^p (\mathbb{R}^{2n}) \) and \( g(x, \xi) = \mathbb{S}_{n,t} (x) \mathbb{S}_{n,t} (\xi), s, t > n/2, \) where \( \mathbb{S}_{n,t} \) is the fundamental solution of the operator \( (1 - \Delta)^{t/2}, \Delta \) is the Laplacian.

A natural question arises: Is it possible to extend Kato's results to a symbol \( a(x_1, x_2, \xi) \) of \( 3n \)-variables? Essentially the question is equivalent to finding a suitable condition under which a symbol \( a(x_1, x_2, \xi) \) of \( 3n \)-variables reduces to a symbol \( a(x, \xi) \) of \( 2n \)-variables.

In [3], Childs proved similar theorems using multiple Hölder continuity of order greater than \( 1/2 \). It is not known that his results still hold for the order of \( 1/2 \).

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References


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