EXTENSIVE SUBCATEGORIES

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Dedicated to Professor Chi Young Kim on his 60th birthday

1. Introduction.

Using limit-operators, we have established in [10] a method to construct new extensive subcategories from well known extensive subcategories in various subcategories of the category \textbf{Haus} of Hausdorff spaces and continuous maps and the category \textbf{HUnif} of Hausdorff uniform spaces and uniformly continuous maps. In this vein, the following question is natural: Can every extensive subcategory containing the known extensive subcategory of the category, be constructed with some limit-operator?

For this question, we introduce new operators which satisfy weaker conditions than limit-operators and still give us same machinery as limit-operators. Moreover, for a hereditary subcategory \( A \) of \textbf{Haus} or \textbf{HUnif} and for an extensive subcategory \( B \) of \( A \), every reflective subcategory of \( A \) containing \( B \) can be characterized with such an operator on \( B \). Also we establish some interesting relationships between those operators and extensive subcategories.

All topological and categorical concepts will be used in the sense of N. Bourbaki [3] and H. Herrlich [7], respectively. In particular, we assume throughout this paper that a subcategory of a category is full and isomorphism-closed.

2. Extensive subcategories.

The category of topological (uniform) spaces and (uniformly resp.) continuous maps will be denoted by \textbf{Top} (\textbf{Unif}, resp.).

2.1 DEFINITION. Let \( B \) be a subcategory of \textbf{Top} or \textbf{Unif}. An operator \( l \) which associates every pair \((X, A)\), where \( X \) is an object of \( B \) and \( A \) is a subset of \( X \), a subset \( lXA \) of \( X \) is said to be an extensive operator on \( B \) if \( l \) satisfies the following conditions:

1) if \( A \) is a subset of \( X \), then \( A \subseteq lXA \subseteq cl_XA \), where \( cl_X \) is the closure operator on \( X \).

2) if \( f: X \rightarrow Y \) is a morphism in \( B \) and \( A \) is a subset of \( X \), then \( f(lXA) \subseteq lYf(A) \).

3) if \( A \) and \( B \) are subsets of \( X \) with \( A \subseteq B \), then \( lXA \subseteq lXB \). An extensive operator \( l \) on \( B \) is said to be idempotent, if \( l \) satisfies the following:

4) if \( A \) is a subset of \( X \in B \), then \( lX(lXA) = lXA \).

It is obvious that every (idempotent) limit-operator (see [8]) is an (idempotent, resp.) extensive operator on \textbf{Top}.

2.2 DEFINITION. Let \( l \) be an extensive operator on \( B \). A subset \( A \) of an object \( X \) of
$B$ is said to be $l$-closed if $l_{X}A = A$. We will denote the family of $l$-closed subsets of $X$ by $S_{l}(X)$.

2.3. For any subcategory $B$ of $\text{Top}$ or $\text{Unif}$, let $E = E(B)$ be the class of all extensive operators on $B$. We define a relation $\leq$ on $E$ as follows: for any pair $(l, l') \in E$, $l \leq l'$ if $l_{X}A \subseteq l_{X}A$ for every $X \in B$ and every subset $A$ of $X$. Then it is easy to show that $(E, \leq)$ becomes a complete “lattice”, where $P(l')$ with $P(l') = \text{cl}_{X}A$ ($l_{X}A = A$, resp.) is the smallest (largest, resp.) element of $E$ and for any subfamily $E'$ of $E$, $(\bigvee \{l | l \in E'\})_{X}A = \bigcap \{l_{X}A | l \in E'\}$ defines the join of $E'$.

For the restricted relation of $\leq$ on the class of $IE(B)$ of all idempotent extensive operators on $B$ again denoted by $\leq$, $(IE(B), \leq)$ is also a complete “lattice” with the same largest and smallest elements, while the join of a subfamily $E'$ of $IE(B)$ in $IE(B)$ is precisely the associated idempotent extensive operator of the meet of the subfamily in $E(B)$.

2.4 REMARK. 1) For any extensive operator $l$ on $B$ there is an associated idempotent extensive operator $l$ on $B$ with $S_{l}(X) = S_{l}(X)$ for every $X \in B$, where $l_{X}A = \bigcap \{B | A \subseteq B \text{ and } B \in S_{l}(X)\}$.

2) The associated idempotent extensive operator $l$ on $B$ of an extensive operator $l$ on $B$ turns out to be the largest idempotent extensive operator on $B$ with $l \leq l'$ and the meet of a subfamily of $IE(B)$ in $IE(B)$ is precisely the associated idempotent extensive operator of the meet of the subfamily in $E(B)$.

3) For any extensive operator $l$ on $B$, there is an associated idempotent limit-operator $l$ on $B$, where for any $X \in B$, $l_{X}$ is defined as the closure operator on $X$ with respect to the topology with $S_{l}(X)$ as a subbase for the closed sets. Furthermore, for any extensive operator $l$ on $\text{Top}$, let $\Theta(l) = \{X \in \text{Top} | \text{ every member of } S_{l}(X) \text{ is closed in } X\}$. Then it is obvious that $\Theta(l) = \Theta(l)$. Hence every extensive operator on $\text{Top}$ generates a coreflective subcategory of $\text{Top}$, for $\Theta(l)$ is a coreflective subcategory of $\text{Top}$(see[8]).

2.5 Definition. Let $A$ be a subcategory of the category $\text{Haus}$ or $\text{HUnif}$. A subcategory $B$ of $A$ is called extensive if it is a reflective subcategory of $A$ such that the $B$-reflection maps $r_{X}: X \longrightarrow rX$ are dense embeddings for each $X \in A$.

It is well known that for every epi-reflective subcategory $B$ of $\text{Haus}$, there is an epi-reflective subcategory $RB$ of $\text{Haus}$ such that $B$ is extensive in $RB$ and for any $X$ in $\text{Haus}$, the $B$-reflection of $X$ is factorized through the $RB$-reflection of $X$ and $B$-reflection of the $RB$-reflection of $X$. Hence every epi-reflective subcategory of $\text{Haus}$ can be completely determined by a certain extensive subcategory in a (hereditary) subcategory of $\text{Haus}$ (see [6]).

Let $B$ be an extensive subcategory $A$ of $\text{Haus}$ or $\text{HUnif}$. For an idempotent extensive operator $l$ on $B$, let $B_{l}$ be the subcategory of $A$ determined by those objects of $A$ which are $l$-closed in their $B$-reflection spaces.

2.6 Theorem. If $A$ is hereditary, then $B_{l}$ is also an extensive subcategory of $A$.

Proof. For every $X \in A$, let $r_{X}: X \longrightarrow rX$ be the $B$-reflection of $X$ such that $X$ is a
dense subspace of \( rX \) and \( r_x \) is the natural embedding. Let \( r_1X \) be the subspace of \( rX \) with \( I_rX \) as its underlying set. Since \( A \) is hereditary, \( r_1X \) belongs to \( A \) and the natural embedding \( r_1X \rightarrow rX \) is a \( B \)-reflection of \( r_1X \). Hence \( r_1X \) is \( I \)-closed in its \( B \)-reflection space \( rX \), so that \( r_1X \) belongs to \( B_I \). By the exactly same arguments as those in [10], we can conclude that the natural embedding \( X \rightarrow rIX \) is a \( B \)-reflection of \( X \).

2.7 Remark. 1) The correspondence \( I \rightarrow B_I \) between \((IE(B), \leq)\) and the class \( Ext_B A \) of all extensive subcategories of \( A \) containing \( B \) with the inclusion relation is monotone.

2) It is well known [2], [9], that for any subspace \( Y \) of the Katětov extension \( sX \) (see [11]) of a Hausdorff space \( X \), \( eX \) and \( eY \) are homeomorphic if \( Y \) contains \( X \). With this and the same argument as that in the above theorem, it is easy to show that the above theorem holds for the case of \( A=\pHaus \) (see [5]) and \( B=\) the subcategory of \( \pHaus \) determined by all \( H \)-closed spaces, i.e. for any idempotent extensive operator \( l \) on the subcategory \( H \) of \( \pHaus \) determined by all \( H \)-closed spaces, the subcategory \( H_I \) of \( \pHaus \) determined by spaces which are \( l \)-closed in their Katětov extensions is also extensive in \( \pHaus \).

3. Reflective subcategories of a Hereditary subcategory of Haus or HUnif.

For a subcategory \( A \) of \( \text{Haus} \) or \( \text{HUnif} \) and an extensive subcategory \( B \) of \( A \), every \( H \)-closed \( A \)-object belongs to \( B \) so that one can easily guess the smallest extensive subcategory of \( A \) containing \( B \). Furthermore, it is easy to show that every reflective subcategory of \( A \) containing \( B \) is also extensive in \( A \).

3.1 Theorem. Let \( A \) be a hereditary subcategory of \( \text{Haus} \) or \( \text{HUnif} \) and \( B \) an extensive subcategory of \( A \). For any reflective subcategory \( E \) of \( A \) containing \( B \), there exists an idempotent extensive operator \( l^E \) on \( B \) with \( E=B\cdot l^E \).

Proof. By the above remark, \( E \) is also extensive in \( A \). For any \( X \) in \( A \), let \( eX : X \rightarrow eX \) be an \( E \)-reflection of \( X \). Let \( A \) be a subset of an object \( X \) of \( B \). Since \( A \) is hereditary, the subspace \( A \) of \( X \) belongs to \( A \). For the natural embedding \( j_A : A \rightarrow X \), there is a unique morphism \( f_A : eA \rightarrow X \) in \( E \) with \( f_A eA = j_A \), for \( X \in B \subseteq E \). We define \( I_A X \) by \( f_A(eA) \). We wish to show that the operator \( l \) defined as the above is an extensive operator on \( B \). Firstly, \( A=j_A(A) \subseteq f_A eA(A) \subseteq f_A(eA)=l_X A \), i.e. \( A \subseteq l_X A \). Moreover, \( I_X A = f_A(eA) = f_A(\text{cl}_A(eA(A))) \subseteq \text{cl}_X f_A(eA(A))) = \text{cl}_X j_A(A) = \text{cl}_X A \), i.e. \( I_X A \subseteq \text{cl}_X A \). Secondly, for any morphism \( h : X \rightarrow Z \) in \( B \), we have the following diagram (Shown in the end of the proof) in which the outer rectangle and the upper trapezoid commute, where \( j_A, e_A \) and \( f_A \) can be understood such as \( j_A, e_A \) and \( h \) is the unique morphism determined by \( e_A \) and \( e_A(A) \) of \( A \). Since \( f_X(A) \) of \( e_A(A) \) of \( A \) is the reflection map. Hence \( h(I_X A) = h(f_X(eA)) = f_{h(X)} h(eA) \subseteq f_{h(A)}(eh(A)) = I_Y h(A), \) i.e. \( h(I_X A) \subseteq I_Y h(A) \). For the condition 3), the proof is simple and left to the reader. Furthermore, for any \( X \in B \), the family \( S_I(X) \) of \( I \)-closed subsets of \( X \) is precisely the family of subsets of \( X \) which belong to \( E \) as subspaces of \( X \). Indeed, suppose a subspace \( X \subseteq B \) does not belong to \( E \). Since \( E \) is extensive in \( A \), \( e_A \) is not onto and \( A \) is dense in \( eA \). Hence \( \phi \neq f_A(eA-A) \subseteq f_A(eA)-f_A(A) = l_X A-A ; \) \( A \) is not \( I \)-closed. It is very simple to show that the other inclusion and the
proof is left to the reader. Let \( l^E \) be the associated idempotent extensive operator on \( B \) with the above extensive operator \( l \). Since \( S_l(X) = S_l^E(X) = \{A \subseteq X | A \in B\} \), it is obvious that \( B^E = E \). This completes the proof.

\[
\begin{array}{ccc}
A & \xrightarrow{\text{h}lA} & h(A) \\
| & \downarrow{j_A} & | \\
eA & \xrightarrow{e_{h(A)}} & eh(A) \\
| & \downarrow{f_A} & | \\
X & \xrightarrow{h} & Y \\
\end{array}
\]

In what follows, let \( A \) and \( B \) be the same categories as those in Theorem 3.1.

3.2 Remark. For any \( E \in \text{Ext}_B A \), \( l^E \) is the largest element of \( IE(B) \) with \( E = B_l \).

Proof. Let \( l' \) be an element of \( IE(B) \) with \( B_l = E \). For any \( X \in B \) and any subset \( A \) of \( X \), we have the following commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{r^A_E} & l'_{r^B A} \\
| & \downarrow{j_A} & | \\
f_A & \xrightarrow{f_A} & f_A \\
| & \downarrow{j_A} & | \\
X & \xrightarrow{l} & X \\
\end{array}
\]

where \( r^B A \) is the \( B \)-reflection of \( A \), \( j_A \) and \( f_A \) are defined as in Theorem 3.1, and \( j_A \) is determined by \( j_A \) and the reflection \( A \rightarrow r^B A \). Then \( l'_{X A} = l'_A f_A (l'r^B A) \subseteq l'_X f_A (A) = l'_X A \), where \( l \) is the extensive operator constructed in Theorem 3.1. Hence \( l' \leq l \). By the Remark 2.4, \( l' \leq l^E \).
3.3 COROLLARY. The correspondence $B \mapsto l^B$ between $\text{Ext}_B A$ and $\text{IE}(B)$ is one-one but not necessarily onto.

Proof. The first assertion is an immediate consequence of Theorem 3.1. For the second part, let $A$ be the category of completely regular spaces and continuous maps, $B$ the category $\text{Comp}$ of compact spaces and continuous maps, and $E$ the category $\text{RComp}$ of real compact spaces and continuous maps. S. Stróńska has shown [13] that there is a completely regular space $M$ which can be represented as the union of two closed subsets $A$, $B$ such that each of them is realcompact in its relative topology and which is not realcompact. Hence $l^n_{x_M} (A \cap B) \neq l^n_{x_M} A \cap l^n_{x_M} B$, where $x_M$ is the Stone–Čech compactification of $M$. Let $\text{cls}_M$ be the $Q$-closure operator (see [12]). Then it is well known [12] that $\text{RComp} = \text{Comp} \circ \text{cls}_M$ and $\text{cls}_M (A \cup B) = \text{cls}_M A \cup \text{cls}_M B$. Hence $\text{cls}_M \neq l^B$.

3.4 REMARK. By the above example, we can also conclude that $B_I$ may contain $E$ properly, where $l$ is the associated idempotent limit–operator on $B$ with $l$.

3.5 THEOREM. The correspondence $l \mapsto B_I$ between $(\text{IE}(B), \leq)$ and $\text{Ext}_B A$ with the inclusion relation preserves arbitrary joins and meets.

Proof. Let $(l^B)_{\lambda \in A}$ be a subfamily of $\text{IE}(B)$. Regarding joins, let $s = \bigvee \{l^A | \lambda \in A\}$. By the remark 2.7, $B_s$ is an upper bound of $\{B_I | \lambda \in A\}$.

For any upper bound $E$ of $\{B_I | \lambda \in A\}$ in $\text{Ext}_B A$, there is an idempotent extensive operator $I^B$ on $B$ with $E = B_I^B$. For any subset $A$ of an object $X$ of $B$ and for any $\lambda \in A$, we have the following commutative diagram

$$
\begin{array}{ccc}
A & \xrightarrow{l^B} & A \\
\downarrow r_{\lambda} & & \downarrow r_{\lambda} \\
X & \xrightarrow{l^A} & X
\end{array}
$$

where $r_{\lambda}, l^B_{r_{\lambda} A}$ and $l^A_{r_{\lambda} A}$ are reflections of $A$ with respect to $B$, $E$ and $B_I$, respectively (also see Theorem 2.6) and $f_{\lambda}, g_{\lambda}$ and $h$ are determined by the reflection property and $B \subseteq B_I \subseteq E$. By the definition of $l$ in Theorem 3.1, $l_{r_{\lambda} A} = f_{\lambda} (l^B_{r_{\lambda} A}) = g_{\lambda} h (l^E_{r_{\lambda} A}) \subseteq g_{\lambda} (l^B \subseteq l^E_{r_{\lambda} A} (A) = l^E_{\lambda A}; I^B \leq l^B$. By the Remark 2.4, $l^B \leq l^E$. Hence $s \leq l^E$, i.e. $B_s \subseteq B_E = E$. Hence $B_s$ is the least upper bound of $\{B_I\}$.

Regarding meets, let $m = \bigwedge \{l^A | \lambda \in A\}$. It is obvious that $B_m$ is a lower bound of
\{B_{\lambda} \mid \lambda \in \Lambda \}. \text{ Let } E \text{ be a lower bound of } \{B_{\lambda} \mid \lambda \in \Lambda \}. \text{ Since } E \subseteq B_{\lambda} \text{ for any } \lambda \in \Lambda, \text{ } \ell_{\alpha} X = X \text{ for every } X \in E; \text{ } m_{\gamma} X = X \text{ (see 2.3). Hence } X \text{ belongs to } B_{\alpha}; \text{ This completes the proof.}

References


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