NORMAL CONDITIONS ON A HYPERSURFACE OF A QUATERNIONIC KAHLERIAN MANIFOLD

BY SANG-SEUP EUM AND UN KYU KIM

§ 0. Introduction.

Recently, K. Yano, U-H. Ki and one of the present authors proved the following [3]

THEOREM A. On a hypersurface of an almost quaternion manifold, the condition
\[ [F,F] + du \wedge U = 0 \] and the condition \[ [G,G] + dv \wedge V = 0 \] and \[ [F,G] + du \wedge V + dv \wedge U = 0 \] are equivalent, where \((F,U,u), (G,V,v)\) and \((H,W,w)\) are the almost contact three structures induced on the hypersurface.

The purpose of the present paper is to prove Theorem A in the case of a hypersurface of a quaternionic Kaehlerian manifold.

§ 1. Preliminaries.

Let \(M^{4n}\) be an almost quaternionic manifold, that is, a 4n-dimensional differentiable manifold which admits a set of three tensor fields \(F, G, H\) of type (1,1) satisfying

\[ F^2 = -I, \quad G^2 = -I, \quad H^2 = -I, \]

\[ \begin{align*}
F &= G'H = -H'G, \quad G = H'F = -F'H, \quad H = F'G = -G'F,
\end{align*} \]

\(I\) denoting the identity tensor.

In a previous paper [3], we proved that there exists a Hermitian metric \(\langle g \rangle\) for the almost quaternionic structure \(F, G, H\), that is, a Riemannian metric \(\langle g \rangle\) satisfying

\[ \langle g \rangle(FX, FY) = \langle g \rangle(X, Y), \]

\[ \langle g \rangle(GX, GY) = \langle g \rangle(X, Y), \]

\[ \langle g \rangle(HX, HY) = \langle g \rangle(X, Y) \]

for arbitrary vector field \(X\) and \(Y\) of \(M^{4n}\). In this case \(M^{4n}\) is called an almost quaternionic metric manifold.

If an almost quaternionic metric manifold \(M^{4n}\) satisfies the condition

\[ \begin{align*}
\langle \gamma \rangle F &= \langle r(X)G - q(X)H \rangle, \\
\langle \gamma \rangle G &= -\langle r(X)F \rangle + \langle p(X)H \rangle, \\
\langle \gamma \rangle H &= \langle q(X)F - p(X)G \rangle,
\end{align*} \]

Received by the editors Aug. 31, 1976.
where \( \nabla \) is the operator of covariant differentiation with respect to \( \gamma \) and \( q, r \) are certain 1-forms, \( X \) being an arbitrary vector field of \( \mathcal{M}^{4n} \), then \( \mathcal{M}^{4n} \) is called a quaternionic K"ahlerian manifold \([1]\).

Suppose that a \((4n-1)\)-dimensional orientable differentiable manifold \( \mathcal{M}^{4n-1} \) is immersed differentiably in an almost quaternionic metric manifold \( \mathcal{M}^{4n} \) by the immersion \( i: \mathcal{M}^{4n-1} \rightarrow \mathcal{M}^{4n} \) and denote by \( B \) the differential of \( i \). We denote by \( C \) the unit normal to \( i(M^{4n-1}) \) with respect to the Hermitian metric \( \gamma \) introduced above.

Then the transform of a vector field tangent to \( i(M^{4n-1}) \) and that of the unit normal vector field by \( F, G \) and \( H \) can be expressed respectively as

\[
\begin{align*}
\nabla_F X &= \nabla_X F + u(X) C, \\
\nabla_G X &= \nabla_X G + v(X) C, \\
\nabla_H X &= \nabla_X H + w(X) C,
\end{align*}
\]

where \( F, G \) and \( H \) are tensor fields of type \((1,1)\), \( U, V, W \) vector fields, \( u, v, w \) 1-forms and \( X \) an arbitrary vector field of \( \mathcal{M}^{4n-1} \).

In the previous paper \([3]\), we proved that \((F, U, u)\), \((G, V, v)\) and \((H, W, w)\) are three almost contact metric structures, that is, they satisfy the following equations.

\[
\begin{align*}
F^2 &= -I + u \otimes U, \\
G^2 &= -I + v \otimes V, \\
H^2 &= -I + w \otimes W,
\end{align*}
\]

\[
\begin{align*}
GH &= F + w \otimes V, \\
HF &= G + u \otimes U, \\
FG &= H + v \otimes V,
\end{align*}
\]

\[
\begin{align*}
\omega F &= 0, & \omega G &= w, & \omega H &= -v, \\
\nu F &= -w, & \nu G &= 0, & \nu H &= u, \\
\omega^2 F &= v, & \omega^2 G &= -u, & \omega^2 H &= 0, \\
FU &= 0, & FV &= W, & FW &= -V,
\end{align*}
\]

\[
\begin{align*}
GU &= -W, & GV &= 0, & GW &= U, \\
HU &= V, & HV &= -U, & HW &= 0,
\end{align*}
\]

\[
\begin{align*}
u(U) &= 1, & u(V) &= 0, & u(W) &= 0, \\
v(U) &= 0, & v(V) &= 1, & v(W) &= 0, \\
w(U) &= 0, & w(V) &= 0, & w(W) &= 1,
\end{align*}
\]

\[
\begin{align*}
g(FX, FY) &= g(X, Y) - u(X) u(Y), \\
g(GX, GY) &= g(X, Y) - v(X) v(Y), \\
g(HX, HY) &= g(X, Y) - w(X) w(Y),
\end{align*}
\]

where \( g \) is the Riemannian metric on \( \mathcal{M}^{4n-1} \) induced from that of \( \mathcal{M}^{4n} \), that is,
Normal conditions on a hypersurface of a quaternionic Kaehlerian manifold

\[ g(BX, BY) = g(X, Y). \]

Let \( \{ U; y^\alpha \} \) and \( \{ U; x^\beta \} \) be the coordinate neighborhoods of \( M^{4n} \) and \( M^{4n-1} \) respectively, where, here and in the sequel, the indices \( \lambda, \mu, \nu, \tau, \ldots \) run over the range \( \{1, 2, \ldots, 4n\} \) and the indices \( h, i, j, k, \ldots \) the range \( \{1, 2, \ldots, 4n-1\} \).

For the hypersurface \( i(M^{4n-1}) \), the equations of Gauss and Weingarten are respectively

\[ \nabla_k B_j^i = h_k^i C^i, \quad \nabla_i C^i = -h_i^i B_i^i, \]

where \( B_j^i = \partial_j y^i \) \( (\partial_j = \partial / \partial x^j) \), \( h_k^i \) is the second fundamental tensor of \( i(M^{4n-1}) \) and \( h_i^i = h_k^k g^{ii} \).

Differentiating (1.4) covariantly along the hypersurface \( i(M^{4n-1}) \) of a quaternionic Kaehlerian manifold and taking account of (1.3) and (1.10), we obtain

\[ \begin{align*}
\nabla_F &= r_k F_j^i - q_k H_j^i - h_k U_j + h_k^i U_j, \\
\nabla G_j^i &= p_k H_j^i - r_k F_j^i - h_k V_j^i + h_k^i V_j, \\
\nabla H_j^i &= q_k F_j^i - p_k G_j^i - h_k W_j^i + h_k^i W_j, \\
\nabla u_j &= r_k v_j - q_k w_j - h_k F_j^i, \\
\nabla v_j &= p_k w_j - r_k u_j - h_k G_j^i, \\
\nabla w_j &= q_k u_j - p_k v_j - h_k H_j^i, 
\end{align*} \]

where \( F_j^i, G_j^i \) and \( H_j^i \) are respectively components of \( F, G \) and \( H, u_j, v_j \) and \( w_j \) those of \( u, v \) and \( w \) and

\[ p_k = 'p_k B_k^i, \quad q_k = 'q_k B_k^i, \quad r_k = 'r_k B_k^i. \]

§ 2. Normal conditions on a hypersurface of a quaternionic Kaehlerian manifold.

In this section, we consider the normal conditions on a hypersurface \( M^{4n-1} \) of a quaternionic Kaehlerian manifold \( M^{4n} \) \( (n \geq 2) \).

The almost contact structure is said to be normal if the tensor

\[ [F, F] + du \otimes U \]

vanishes, where \([F, F]\) is the Nijenhuis tensor formed with \( F \).

We compute components of this tensor.

\[ [F, F]_j^h + (\nabla_j u_i - \nabla_i u_j) U^h \]

\[ = (r_i F_j^i - q_i) G_j^h - (r_i F_j^i - q_i) G_j^h \]

\[ - (q_i F_j^i + r_i) H_j^h + (q_i F_j^i + r_i) H_j^h \]

\[ + (F_j^i h_j^h - h_j^i F_j^h) u_i - (F_j^i h_j^h - h_j^i F_j^h) u_j, \]

Similarly, computing components of the tensors
respectively, we find

\[ [G, G]_j^h + (p_j v_i - p_i v_j) V^h \]

\[ (2.2) \]

\[ = (p_j G_j^i - r_j) H^h_i - (p_i G_i^j - r_i) H^h_j \]

\[ - (r_j G^j_i + p_j) F^h_i + (r_i G^i_j + p_i) F^h_j \]

\[ + (G^j_i h^h_i - h^h_i G^h_j) v_i - (G^i_j h^h_j - h^h_j G^h_i) v_j, \]

\[ [H, H]_j^h + (q_j w_i - p_i w_j) W^h \]

\[ (2.3) \]

\[ = (q_j H_j^i - r_j) F^h_i - (q_i H_i^j - p_i) F^h_j \]

\[ - (p_j H^j_i + q_j) G^h_i + (p_i H^i_j + q_i) G^h_j \]

\[ + (H^j_i h^h_i - h^h_i H^i_j) w_i - (H^i_j h^h_j - h^h_j H^j_i) w_j. \]

We also compute components of the tensor

\[ [F, G]_j^h + d \omega \otimes V + d \nu \otimes U. \]

\[ [F, G]_j^h + (p_j u_i - p_i u_j) V^h + (q_j v_i - q_i v_j) U^h \]

\[ (2.4) \]

\[ = (p_j F_j^i - q_j G_j^i) H^h_i - (p_i F^i_j - q_i G^i_j) H^h_j \]

\[ + (r_j F^j_i + p_j) G^h_i - (r_i F^i_j + p_i) G^h_j \]

\[ - (r_j F^j_i - q_j) F^h_i + (r_i F^i_j - q_i) F^h_j \]

\[ + (F^j_i h^h_i - h^h_i F^i_j) v_i - (F^i_j h^h_j - h^h_j F^j_i) v_j \]

\[ + (G^j_i h^h_i - h^h_i G^i_j) u_i - (G^i_j h^h_j - h^h_j G^j_i) u_j. \]

Suppose that the almost contact structures \((F, U, u)\) and \((G, V, v)\) are both normal. In this case, contracting with respect to \(j\) and \(h\) in (2.1), we find

\[ v_i r_i U^h - w_i q_i U^h - F^h_i h^h_i u_i = 0. \]

Transvecting (2.5) with \(V^i\) and \(W^i\) respectively, we find

\[ r_i U^h = h^h_i W^h u_i = h^h_i U^h W^i, \]

\[ (2.6) \]

\[ q_i U^i = h^h_i V^h u_i = h^h_i U^h V^i. \]

Transvecting (2.5) with \(F^i_j\), we find

\[ h^h_i u_i - h^h_i U^h u_i = w_j r_i U^i - v_j q_i U^i = 0. \]

Transvecting (2.8) with \(V^k\), we find

\[ h^h_i U^i V^i = q_i U^i. \]

\[ (2.9) \]
Transvecting (2.1) with $G_h^k$ and contracting with respect to $i$ and $k$, we find

\[(2.10) \quad (-4n+4) (r_i F^j - q_j) - (r_i W^j - q_i V^j) v_j + q_i U^j w_j + (q_i W^j + r_i V^j) w_j
\]

\[+ F^j h_i^j w_j - h_i^j v_i = 0.\]

Transvecting (2.10) with $U^j$, we find

\[(2.11) \quad (4n-3) q_i U^j = h_i^j V^j v_i = h_i^j U^j V^i.\]

Comparing (2.11) with (2.7), we have

\[(2.12) \quad q_i U^j = 0, \quad h_i^j U^j V^i = 0.\]

Transvecting (2.10) with $W^j$, we find

\[(2.13) \quad (4n-3) (r_i U^j + q_i W^j) - 2 h_i^j V^i W^j = 0.\]

On the other hand, contracting with respect to $j$ and $h$ in (2.2), we find

\[(2.14) \quad w_i p_i V^i - r_i V^i - G_i h_i^j v_j = 0.\]

Transvecting (2.14) with $W^i$ and taking account of (2.12), we have

\[(2.15) \quad p_i V^i = 0.\]

Transvecting (2.14) with $U^j$, we find

\[(2.16) \quad r_i V^i = h_i^j W^j v_i = h_i^j V^i W^i.\]

Transvecting (2.2) with $H_h^k$ and contracting with respect to $i$ and $k$, we find

\[(2.17) \quad (-4n+4) (p_i G^j - r_j) + (r_i W^j - p_i U^j) w_j + r_i V^j v_j + (r_i U^j + p_i W^j) u_j
\]

\[+ G_j h_i^j u_i - h_j^j w_i = 0.\]

Transvecting (2.17) with $V^i$, we find

\[(2.18) \quad (4n-3) r_i V^i = h_i^j V^i w_i = h_i^j V^i W^i.\]

Comparing (2.18) with (2.16), we have

\[(2.19) \quad r_i V^i = 0, \quad h_i^j V^i W^i = 0\]

Transvecting (2.10) with $W^j$ and taking account of (2.19), we have

\[(2.20) \quad q_i W^i = 0.\]

Transvecting (2.17) with $U^j$, we find

\[(2.21) \quad r_i U^j + p_i W^j - 2 h_i^j W^j u_i = 0.\]

Transvecting (2.10) with $V^i$, we find
(2.22) \((-4n+3)(r \sigma W' - q\alpha V') + h_{\alpha} W' W' - h_{\alpha} V' V' = 0.\)

Transvecting (2.17) with \(W^i\), we find

(2.23) \((-4n+3)(p t U^i - r t W^i) + h_{\alpha} U^i U^i - h_{\alpha} W^i W^i = 0.\)

Making (2.22) \(+ (2.23)\), we find

(2.24) \((-4n+3)(p t U^i - q t V^i) + h_{\alpha} U^i U^i - h_{\alpha} V^i V^i = 0.\)

Substituting (2.19) \(+ (2.20)\) into (2.13), we have

(2.25) \(r t W^i = 0.\)

Substituting (2.12) \(+ (2.25)\) into (2.5), we find

(2.26) \(F^i h^j u = 0.\)

Transvecting (2.26) with \(F^i\), we have

(2.27) \(h^i u = \lambda u, \quad \lambda = h_{\alpha} U^i U^i.\)

Substituting (2.15) \(+ (2.19)\) into (2.14), we find

(2.28) \(G^i h^j v = 0.\)

Transvecting (2.28) with \(G^i\), we have

(2.29) \(h^i v = \mu v, \quad \mu = h_{\alpha} V^i V^i.\)

Taking account of (2.25) \(+ (2.6)\), we have from (2.21)

(2.30) \(p t W^i = 0, \quad h_{\alpha} U^i W^i = 0.\)

Transvecting (2.1) with \(V^i\), we find

(2.31) \((-r t W^i + q t V^i) G^i + (q t F^j v^j) U^i + (-W^i h_{\alpha} + \mu W^i) u = 0.\)

Transvecting (2.31) with \(U^i\) and taking account of (2.29), we find

(2.32) \((r t W^i - q t V^i) W^i - (W^i h_{\alpha} - \mu W^i) = 0.\)

Transvecting (2.32) with \(w_{\alpha}\), we find

(2.33) \(r t W^i - q t V^i = h_{\alpha} W^i W^i - \mu.\)

Comparing (2.33) with (2.22), we have

(2.34) \(r t W^i = q t V^i, \quad \mu = \nu, \quad \nu = h_{\alpha} W^i W^i.\)

Substituting (2.34) into (2.32), we find

(2.35) \(h^i_{\alpha} w = \mu w.\)
Transvecting (2.2) with $W^i$, we find

$$(-p_1U^i + r_1W^i)H_j^k + (r_1G_j^i + \rho_j)V^k - (U^i_h^k - U^k)\nu_j = 0.$$  

Transvecting (2.2) with $W^iV^j\nu_k$ and taking account of (2.24) and (2.34), we have

$$p_1U^i = r_1W^i, \quad \lambda = \mu.$$  

Gathering above results, we have

$$(2.37) \quad p_1U^i = q_1V^i = r_1W^i, \quad p_1V^i = p_1W^i = q_2U^i = q_2W^i = r_1U^i = r_1W^i = 0.$$  

$$h_1U^iU^j = h_1V^iV^j = h_1W^iW^j.$$  

Therefore we have from (2.31) and (2.36),

$$(2.39) \quad q_1F^i = -r_1, \quad r_1G^i = -\rho_1.$$  

by virtue of (2.27), (2.35), (2.37) and (2.38).

Transvecting the first equation of (2.39) with $F^i_k$, $G^i_k$ and $H^i_k$ respectively and taking account of (2.37), we have

$$(2.40) \quad q_1 = r_1F^i_k, \quad q_1H^i_k = -r_1G^i_k, \quad q_1G^i_k = r_1H^i_k.$$  

Similarly from the second equation of (2.39) we have

$$(2.41) \quad r_1H^i_k = p_1F^i_k, \quad r_1 = p_1G^i_k, \quad r_1F^i_k = -p_1H^i_k.$$  

Combining (2.39), (2.40) and (2.41), we obtain

$$(2.42) \quad p_1F^i_k = q_1G^i_k = r_1H^i_k,$$

$$(2.43) \quad p_1 = q_1H^i_k = -q_1G^i_k, \quad q_1F^i_k = -r_1H^i_k, \quad r_1G^i_k = -p_1F^i_k.$$  

On the other hand, transvecting (2.1) with $U^i$ and taking account of (2.43) and (2.27), we easily see that

$$(2.44) \quad F^i_jh^j_k - h^i_jF^j_k = 0.$$  

Similarly, transvecting (2.2) with $V^i$ and taking account of (2.43) and (2.29), we also see that

$$(2.45) \quad G^i_jh^j_k - h^i_jG^j_k = 0.$$  

Transvecting (2.44) with $G^i_j$ and taking account of (2.45), (2.27) and (2.29), we easily see that

$$(2.46) \quad H^i_jh^j_k - h^i_jH^j_k = 0.$$  

Substituting (2.43) and (2.46) into (2.3), we conclude that the almost contact structure $(H, W, \nu)$ also is normal.

Thus we have the following
THEOREM 2.1. Let \((F, U, u)\), \((G, V, v)\) and \((H, W, w)\) are the almost contact three structures induced on a hypersurface of a quaternionic Kaehlerian manifold \(\mathcal{M}^{4n}\) \((n \geq 2)\). If two of the structures are normal, then the other also is normal.

§ 3. Normal conditions on a hypersurface of a quaternionic Kaehlerian manifold.

(continued)

In this section we consider the inverse case of §2. We also assume that the dimension of a quaternionic Kaehlerian manifold is \(4n\) and \(n \geq 2\).

If the almost contact structures \((F, U, u)\) and \((G, V, v)\) are both normal, then the equations (2.42), (2.43), (2.44) and (2.45) are satisfied. Substituting these equations into (2.4), we see that

\[
[F, G] \mu^k + (\nu \mu_i - \nu \mu_j) V^k + (\nu \nu_i - \nu \nu_j) U^k = 0.
\]

Conversely, suppose that two almost contact structures \((F, U, u)\) and \((G, V, v)\) satisfy (3.1). In this case, contracting with respect to \(j\) and \(h\) in (3.1) and taking account of (2.4), we find

\[
(p_i U^i - q_i V^i) w_i + r_i V^i v_i - r_i U^i u_i - F_i^h v_i - G_i^h u_i = 0.
\]

Transvecting (3.2) with \(U^i\) and \(V^i\) respectively, we find

\[
r_i U^i = h_i^i u_i W^i = h_i^i U^i W^i, \quad r_i V^i = h_i^i v_i W^i = h_i^i V^i W^i.
\]

Transvecting (3.2) with \(W^i\), we find

\[
p_i U^i - q_i V^i + h_i^i V^i V^i - h_i^i U^i U^i = 0.
\]

Transvecting (3.1) with \(U^i\) and taking account of (2.4), we find

\[
(p_i F_j^i - q_i G_j^i) V^k - (r_i G_j^i + p_i) W^k - q_i W^i H_j^k + (r_i W^i - p_i U^i) G_j^k - q_i U^i F_j^i + h_i^i U^i F_j^i v_j + (W^k h_i^k + h_i^i U^i G_j^k) u_j + G_j^k h_i^k - h_j^k G_i^k = 0.
\]

Transvecting (3.5) with \(v_k\), we find

\[
p_i F_j^i - q_i G_j^i - q_i W^i u_j + q_i U^i w_j - h_i^i U^i w v_j + G_j^i h_i^i v_i + W^i h_i^i v_i u_j = 0.
\]

Transvecting (3.6) with \(V^i\), we find

\[
p_i W^i = r_i U^i = h_i^i U^i W^i
\]

by virtue of (3.3).

Transvecting (3.6) with \(W^i\), we find

\[
p_i V^i = h_i^i U^i V^i.
\]

Transvecting (3.1) with \(H_i^k\), taking account of (2.4) and contracting with respect to \(i\) and \(k\), we find
Normal conditions on a hypersurface of a quaternionic Kaehlerian manifold 175

\[ (-4n+4) (p_t F^j - q_t G^j) + (p_t V^j + q_t U^j) w_j - p_t W^j v_j - q_t W^j u_j \]

(3.9)

\[ + F^j h_t^j u_t - G^j h_t^j v_t = 0. \]

Transvecting (3.9) with \( V^j \), we find

(3.10)

\[ (4n-3) p_t W^t = h_t^t u_t W^t = h_t u_t U^t W^t. \]

Comparing (3.10) with (3.7), we have

(3.11)

\[ p_t W^t = 0, \quad r_t U^t = 0, \quad h_t U^t W^t = 0. \]

Transvecting (3.9) with \( W^j \), we find

(3.12)

\[ (4n-3) (p_t V^j + q_t U^j) - 2 h_t^t V^t u_t = 0. \]

Transvecting (3.9) with \( U^t \), we find

(3.13)

\[ (4n-3) q_t W^t - h_t V^t W^t = 0. \]

Transvecting (3.1) with \( V^t \) and taking account of (2.4), we find

(3.14)

\[ \begin{align*}
- (p_t F^j - q_t G^j) U^j - (r_t F^j - q_j) W^k - q_t V^j G^k + (r_t W^t - q_t V^t) F^j \\
- (W^h h^k - h_t^k V^j F^j) v_j + h_t^j V^j G^k u_j + F^j h_t^k - h_t h^j F^j = 0.
\end{align*} \]

Transvecting (3.14) with \( U^j \), we find

(3.15)

\[ - q_t W^t U^h + (p_t V^j + q_t U^j) W^k - h_t^t U^t F^h + h_t^j V^j G^h = 0. \]

Transvecting (3.15) with \( u_h \), we find

(3.16)

\[ q_t W^t - h_t V^t W^t = 0. \]

Comparing (3.16) with (3.13) and taking account of (3.3), we have

(3.17)

\[ q_t W^t = 0, \quad r_t V^t = 0, \quad h_t U^t W^t = 0. \]

Transvecting (3.15) with \( w_h \), we find

(3.18)

\[ p_t V^t + q_t U^t - 2 h_t U^t V^t = 0. \]

Comparing (3.12) with (3.18) and taking account of (3.8), we have

(3.19)

\[ p_t V^t = 0, \quad q_t U^t = 0, \quad h_t U^t V^t = 0. \]

Transvecting (3.14) with \( v_h \), we find

(3.20)

\[ \begin{align*}
- r_t W^j - q_t V^t \end{align*} w_j + F^j h_t^j v_h + h_t w_i = 0. \]

Transvecting (3.20) with \( W^j \), we find

(3.21)

\[ q_t V^t - r_t W^t - h_t V^j W^j + h_t W^t W^j = 0. \]
Transvecting (3.1) with $G_k^i$, taking account of (2.4) and contracting with respect to $i$ and $k$, we find

\[(3.22) \quad (-4n+4) (r_i G_j^i + p_j) - (p_i U^i - r_i W^i) u_j + G_j^i h_j^i w_i + h_j^i u_i = 0.\]

Transvecting (3.22) with $U^i$, we find

\[(3.23) \quad (-4n+3) (p_i U^i - r_i W^i) + h_{is} U^i U^s - h_{is} W^i W^s = 0.\]

Transvecting (3.1) with $F_k^i$, taking account of (2.4) and contracting with respect to $i$ and $k$, we find

\[(3.24) \quad (-4n+4) (r_i F_j^i - q_j) + (q_i V^i - r_i W^i) v_j + F_j^i h_j^i w_i - h_j^i v_i = 0.\]

Transvecting (3.24) with $V^j$, we find

\[(3.25) \quad (-4n+3) (r_i W^i - q_j V^j) + h_{is} W^i W^s - h_{is} V^i V^s = 0.\]

Making (3.23) + (3.25), we find

\[(3.26) \quad (-4n+3) (p_i U^i - q_j V^j) + h_{is} U^i U^s - h_{is} V^i V^s = 0.\]

Comparing (3.26) with (3.4), we find

\[(3.27) \quad p_i U^i = q_i V^i, \quad h_{is} U^i U^s = h_{is} V^i V^s.\]

Substituting (3.27) into (3.23) we find

\[(3.28) \quad (-4n+3) (q_i V^i - r_i W^i) + h_{is} V^i V^s - h_{is} W^i W^s = 0.\]

Comparing (3.21) and (3.28), we find

\[(3.29) \quad q_i V^i = r_i W^i, \quad h_{is} V^i V^s = h_{is} W^i W^s.\]

Transvecting (3.20) with $F_k^i$, we find

\[(3.30) \quad -h_j^i v_i + F_k^j h_j^i w_i = 0\]

by virtue of (3.29).

Substituting (3.30) and (3.29) into (3.24), we have

\[(3.31) \quad q_j = r_i F_j^i.\]

Substituting (3.19) into (3.15), we find

\[(3.32) \quad F_j^i h_j^i w_i - G_j^i h_j^i v_i = 0.\]

Substituting (3.11), (3.17), (3.19) and (3.32) into (3.9), we find

\[(3.33) \quad p_i F_j^i - q_i G_j^i = 0.\]

Transvecting (3.33) with $F_k^i$, we find
Normal conditions on a hypersurface of a quaternionic Kaehlerian manifold

(3.34) \[ p_j = q_i H^j_i. \]

Transvecting (3.31) with \( F_k^j \), we find

(3.35) \[ r_j = -q_i F^j_i. \]

Transvecting (3.34) with \( H_k^j \), we find

(3.36) \[ q_j = -p_i H^j_i. \]

Transvecting (3.33) with \( H_k^j \) and taking account of (3.35), we find

(3.37) \[ r_j = p_i G^j_i. \]

Transvecting (3.37) with \( G_k^j \), we find

(3.38) \[ p_j = -r_i G^j_i. \]

Substituting (3.20), (3.31) and (3.33) into (3.14), we find

(3.39) \[ h_i^j V^k G^i_k u_j + F_j^i h_k^j - h_j^i F_k^h = 0. \]

Substituting (3.17), (3.19) and (3.33) into (3.6), we find

(3.40) \[ G_j^i h^j_{i\nu} = 0. \]

Substituting (3.40) into (3.39), we have

(3.41) \[ F_j^i h_k^h - h_j^i F_k^h = 0. \]

Substituting (3.40) into (3.32), we find

(3.42) \[ F_j^i h_i^k u_s = 0. \]

Substituting (3.20) into (3.2), we find

(3.43) \[ G_j^i h_i^s u_s = h_j^i u_s. \]

Substituting (3.42) and (3.43) into (3.5), we have

(3.44) \[ G_j^i h_i^h - h_j^i G_k^h = 0. \]

Substituting (3.31), (3.35) and (3.41) into (2.1), we obtain

\[ [F, F]_{ij}^k + (\nu_j u_i - \nu_i u_j) U^k = 0. \]

Substituting (3.37), (3.38) and (3.44) into (2.2), we obtain

\[ [G, G]_{ij}^k + (\nu_j v_i - \nu_i v_j) V^k = 0. \]

Thus we have the following
THEOREM 3.1. On a hypersurface of a quaternionic Kaehlerian manifold \(M^{4n}(n\geq 2)\), the condition

\[[F, F] + du \otimes U = 0 \quad \text{and} \quad [G, G] + dv \otimes V = 0\]

and the condition

\[[F, G] + du \otimes V + dv \otimes U = 0\]

are equivalent, where \((F, U, u), (G, V, v)\) and \((H, W, w)\) are the almost contact three structures induced on the hypersurface.

References


Sung Kyun Kwan University

Kook Min University