ON INJECTIVE RINGS

BY JAE-MYUNG CHUNG

1. Introduction

In [1] E. P. Armendariz and S. A. Steinberg concerned regular self-injective rings with a polynomial identity. Since the maximal quotient ring of a ring with zero singular ideal is von Neumann regular and self-injective, this paper investigates the structure of a biregular self-injective ring $R$ by looking at its prime ideals, essential ideals or closed ideals. The reader is referred to [2] for the definitions and basic properties.

Throughout this paper we assume that $R$ is a biregular self-injective ring with unit and $C$ is the center of $R$.

2. Properties of $R$

PROPOSITION 1. Let $A$ be an ideal of $R$. Then $A$ is essential in $R$ if and only if $\text{Ann}_t(A) = (0)$.

Proof. If $A$ is essential in $R$ and $x \in \text{Ann}_t(A)$, then $xR = (0)$. Assume $xR \neq (0)$. Then $xR \cap A \neq (0)$ and there is a nonzero element $a$ in $A$ such that $a = xr$ for some $r \in R$. Since $R$ is semprime and $aRa = xRar \subseteq xA = (0)$, we have $a = 0$. This contradicts the fact that $a \neq 0$. Therefore $xR = 0$ and this yields that $\text{Ann}_t(A) = (0)$. Conversely, if $I$ is a nonzero right ideal of $R$, then $IA \neq (0)$. Since $IA \subseteq I \cap A$, $A$ is essential in $R$.

PROPOSITION 2. If $I$ is a nonzero right ideal of $R$, then $I \cap C \neq (0)$

Proof. Let $x$ be a nonzero element of $I$. Then $xR = eR$ for some central idempotent $e$ since $R$ is biregular. Hence $e \in I \cap C$.

PROPOSITION 3. Let $I$ be an essential right ideal of $R$. Then $I$ contains an essential two-sided ideal of $R$.

Proof. Let $U$ be the largest two-sided ideal of $R$ which is contained in $I$. 

Suppose $U$ is not an essential ideal of $R$. Then $V = \text{Ann}_I(U) \neq (0)$ and $(UV)^\# = U(VU) = (0)$. Since $R$ is semiprime, $U V = (0)$. Let $W = VR$. By proposition 2, $I \cap W \cap C$ contains a nonzero element $w$. Then $w \in U$ and $U \subseteq U + wR \subseteq I$.

**Proposition 4.** Let $A$ be an ideal of $R$ and $J$ an ideal of $C$. Then (a) $(A \cap C) R = A$ and (b) $JR \cap C = J$.

**Proof.** For any $a \in A$, $aR = eR$ for some central idempotent $e$. Since $a = er$ for some $r \in R$, $a = er \in (A \cap C) R$ and this proves (a). Let $c \in JR \cap C$. Then $c = \sum j_t r_t$, $j_t \in J$, $r_t \in R$. Since $C$ is regular, $\sum j_t C = eC$ for some idempotent $e$ contained in $J$. Thus $c = ec \in J$ and (b) is proved.

**Proposition 5.** Let $P$ be an ideal of $R$ and $D$ an ideal of $C$. Then (a) $P$ is prime in $R$ if and only if $P \cap C$ is prime in $C$; (b) $D$ is prime in $C$ if and only if $DR$ is prime in $R$.

**Proof.** (a) Suppose $P$ is prime in $R$ and $E$ and $F$ are ideals of $C$ such that $EF \subseteq P \cap C$. Then $(ER)(FR) = (EF) R \subseteq (P \cap C) R = P$. Therefore $ER \subseteq P$ or $FR \subseteq P$ and this implies $E \subseteq P \cap C$ or $F \subseteq P \cap C$. Conversely, let $A$ and $B$ be ideals of $R$ such that $AB \subseteq P$. Since $P \cap C$ is prime in $C$ and since $(A \cap C) \subseteq AB \cap C \subseteq P \cap C$, $A \cap C \subseteq P \cap C$ or $B \cap C \subseteq P \cap C$. Hence $A \subseteq P$ or $B \subseteq P$. Similarly (b) can be proved.

**Proposition 6.** Let $A$ be an ideal of $R$ and $J$ an ideal of $C$. Then (a) $A$ is closed in $R$ if and only if $A \cap C$ is closed in $C$; (b) $J$ is closed in $C$ if and only if $JR$ is closed in $R$.

**Proof.** Suppose $A \cap C$ is closed in $C$ and $B$ is an essential extension of $A$. For any $t \in B \cap C$ such that $t(B \cap C) \neq (0)$, there exists $a \in A$ such that $a = tb$ or some $b \in B$ since $t B \cap A \neq (0)$. Since $b r = e r$ for some central idempotent $e$ of $R$, $e = b s$ for some $s \in R$ and $a = e u$ for some $u \in R$. Then $0 \neq as = te \in t(B \cap C) \cap (A \cap C)$ and this means that $B \cap C$ is an essential extension of $A \cap C$. Since $A \cap C$ is closed in $C$, $A \cap C = B \cap C$ and $A = B$. This proves that $A$ is closed in $R$. Conversely let $D$ be an essential extension of $A \cap C$. For any $t \in DR$ such that $tDR \neq (0)$, $tDR = dDR$ for some $d \in D$ and $tDR \cap A = d DR \cap (A \cap C) \neq (0)$. Then $DR$ is an essential extension of $A$ and this implies $DR = A$ since $A$ is closed in $R$. Hence $D = A \cap C$ and $A \cap C$ is closed in $C$. Similarly we can prove (b).
COROLLARY (a) A is essential in R if and only if $A \cap C$ is essential in C;
(b) J is essential in C if and only if $JR$ is essential in R.

We consider the set $L(R)$ of closed ideals of R and the set $L(C)$ of closed ideals of C. From proposition 4 and proposition 6, there is a 1-1 correspondence between $L(R)$ and $L(C)$ given by $A \mapsto A \cap C$, $J \mapsto JR$ where $A \in L(R)$ and $J \in L(C)$. Similarly there is a 1-1 correspondence between the essential ideals of R and the essential ideals of C, and there is a 1-1 correspondence between prime ideals of R and prime ideals of C. With an eye to the commutative theory, a biregular self-injective ring with unit can be characterized by the properties of closed, prime or essential ideals. Moreover a prime ideal in R is either essential or closed [3].

References


Seoul National University