CHAIN CONDITIONS AND Q-MODULES

BY WUHAN LEE

It will be assumed that all rings have an identity and that the modules are unital. Modules will be right $R$-modules, and homomorphisms will be $R$-homomorphisms unless otherwise stated. Previously the author defined a $q$-module to be an injective module in which every submodule is quasi-injective and obtained several characterizations of a $q$-modules and investigated the endomorphism ring of a $q$-module [4].

Later E. Lee [5] established that a left $S$-submodule $sN$ of $M_R$, $S=\text{Hom}_R(N,N)$ is noetherian if and only if $N_R$ is noetherian with respect to annihilator submodules for subsets in $R$. Further, he studied that if $R$ is a right artinian ring and $N_R$ a submodule of a $q$-module $M_R$ then the left $S$-module $sN$ is noetherian. Now the purpose of this paper is to study properties of a $q$-module with chain conditions over a commutative ring.

Let $R$ be any ring (not necessarily commutative) and $M$ a right $R$-module. Put $S=\text{Hom}_R(M,M)$, then we assume that $M$ is a left $S$-module. Let $N$ be a subset of $M$. Then we denote the annihilator ideal of $N$ in $S$ and in $R$ by $l(N)$ and $\text{ann} N$, respectively. Similarly, by $r(A)$ we denote the annihilator submodule of $M$ for a left ideal $A$ in $S$. We call $M$ a weakly distinguished $R$-module if for any $R$-submodules $N_1 \supseteq N_2$ in $M$ such that $N_1/N_2$ is $R$-irreducible, $\text{Hom}_R(N_1/N_2, M)=0$. If $M$ is quasi-injective then $M$ is weakly distinguished if and only if $rl(N)=N$ for any $R$-submodule $N$ in $M$ [2, Proposition 6].

Finally, we shall assume that a ring $R$ is commutative. In his paper [1] Harada states that if $R$ is a commutative ring and $M$ is a noetherian quasi-injective module then $S=\text{Hom}_R(M,M)$ is left and right artinian. Since every submodule of a $q$-module is quasi-injective and injectivity of $M_R$ implies that of quasi-injectivity, the following statement is immediate.

**Proposition 1.** Let $R$ be a commutative ring and $M_R$ is a noetherian $q$-module.
If \( N \) is a submodule of \( M \) then \( S=\text{Hom}_R(N,N) \) is left and right artinian.

**Proof.** Since every submodule of a noetherian module is again noetherian the result is evident by [2, Theorem 1].

Let \( P \) be a prime ideal in a commutative ring \( R \). And let \( E(R/P) = E \) be an injective hull of \( R/P \). Then Matlis showed in [7] that \( E = \bigcup A_i \) and \( \text{Hom}_R(E,E) \) is a complete local noetherian ring, where \( A_i = \{ x \in E \mid xP^i = 0 \} \). Let \( \{ P_i \} \) be a finite set of distinct maximal ideals in \( R \). Then according to Harada [1], every \( R \)-submodule \( N \) of \( \bigoplus E(R/P) \) is weakly distinguished and quasi-injective.

Since its implication seems to be interesting, we furnish a rough proof here.

**Proof.** We may assume that \( N \) is an essential submodule of \( E = \bigoplus E_i \), \( E_i = E(R/P) \). Then \( \text{ann} \ x \supset P_i \) for any \( x \) in \( N \). Let \( N_1, N_2 \) be \( R \)-submodules of \( N \) such that \( N_1/N_2 \) is \( R \)-irreducible, then \( N_1/N_2 \cong R/P_i \) for some \( P_i \). Since \( N \cap R/P_i \neq (0) \), \( \text{Hom}_R(N_1/N_2, N) \neq (0) \), which means that \( N \) is weakly distinguished. Hence, \( E \) is an \( R \)-weakly distinguished injective module. Moreover, if we put \( S = \text{Hom}_R(E,E) \), then \( S = \text{Hom}_R(E,E) \). Hence, every \( R \)-submodule \( N \) is an \( S \)-submodule by [1, Lemma 1]. Let \( E' \) be an injective hull of \( N \) contained in \( E \). Then \( E = E' \oplus E'' \) and \( E' \supset N \). \( S' = \text{Hom}_R(E', E') \) may be regarded as a subring of \( S \). Hence, \( M \) is also an \( S' \)-module. Therefore, \( N \) is \( R \)-quasi-injective by [3, Theorem 1.1].

**Proposition 2.** Let \( R \) be a commutative noetherian ring and \( \{ P_i \} \) be a finite set of distinct maximal ideals in \( R \). Then the direct sum of injective hulls \( \bigoplus E(R/P) \) is a weakly distinguished \( q \)-module.

**Proof.** The injective hulls are naturally injective and hence the conclusion is immediate from the definition of a \( q \)-module.

Now assume that a ring \( R \) is not necessary commutative. A. Koehler [4] obtained a characterization for quasi-injective modules over left artinian rings which have a finitely generated, lower distinguished (contains an isomorphic copy of every simple module), and injective module \( Q \). This class of rings includes quasi-Frobenius rings and finitely generated algebras over commutative artinian rings. According to Koehler, a module \( M_R \) over such a ring is quasi-injective if and only if

\[
M = \bigoplus_{i=1}^{k} \left( \text{Hom}_R(e_i S/e_i J, Q) \right)^{\times (i)}
\]
where $S=\text{Hom}_R(Q,Q)$, $e_i$ is an indecomposable idempotent in $S$ for $i=1, \ldots, k$, $J$ is an ideal of $S$, the number of nonisomorphic simple $R$-modules is $k$, and for $i \neq j$, $e_iS \not\cong e_jS$. This decomposition is unique up to automorphism. Here $\sum \oplus M_i^{g(i)}$ denotes the $g(i)$ copies of $M$ and $g(i)$ can be any cardinal number. If $g(i)=0$, then $M_i^{g(i)}=0$.

**Proposition 3.** Let $R$ be a left artinian ring and have a finitely generated lower distinguished, and injective module $Q$. Then a submodule $N_R$ of a $q$-module $M_R$ is expressed uniquely (up to automorphism) as

$$N = \sum_{i=1}^{k} \oplus (\text{Hom}_R(e_iR/e_iJ, R))^{g(i)}$$

where $S=\text{Hom}_R(Q,Q)$, $e_i$ is an indecomposable idempotent in $S$ for $i=1, \ldots, k$, $J$ is an ideal of $S$, the number of nonisomorphic simple $R$-modules is $k$, and for $i \neq j$, $e_iS \not\cong e_jS$.

**Proof.** Obvious.

**Corollary.** Let $R$ be quasi-Frobenius. Then a submodule $N_R$ of a $q$-module $M_R$ is expressed uniquely (up to automorphism) as

$$N = \sum_{i=1}^{k} \oplus (\text{Hom}_R(e_iR/e_iJ, R))^{g(i)}$$

**Proof.** $R$ being quasi-Frobenius implies $R$ is left artinian, selfinjective lower distinguished, and finitely generated. Also $R=\text{Hom}_R(R, R)$.

**References**


Seoul National University