DUAL GROUPS OF LOCALLY COMPACT ABELIAN GROUPS

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The theory of topological groups has been studied by many mathematicians, for example, Ellis [1], Husain [2], Pontrjagin [3], etc. In particular, dual groups of topological groups are studied by Wu [4] and Bourbaki.

In this paper, we shall study about Hausdorff locally compact abelian topological groups. In order to do this, in § 1 we shall prove a proposition about S-topology (Proposition 1.1), in § 2 we verify that the dual group of a Hausdorff locally compact abelian topological group is Hausdorff, locally compact and abelian (Theorem 2.6) using our lemmas 2.3 and 2.4 and in § 3. We prove that (a) the dual group of a compact Hausdorff abelian topological group is a discrete abelian topological group, (b) the dual group of a discrete abelian topological group is compact (Theorem 3.1) and illustrate some examples for Theorem 3.1 (Examples 3.2~3.4).

Throughout this paper we shall assume that G is a topological group without any statements.

§ 1. S-topologies

Let G be a topological group with topology ℱ. Then $G^G = \prod_{g \in G} G$, where $G_x$ is equal to $G_x$ is a topological group with product topology. The product topology has the subbase which is the set of all

$$T(\{g\}, U) = \{ f \in G^G | f(g) \in U \}$$

where $g \in G$ and $U$ an open subset of $G$, i.e., $U \in \mathcal{U}$.

Let $S$ is the set of all closed subsets of $G$. That is, $F \in S$ if and only if the complement of $F (=F^c)$ is in $\mathcal{U}$. Consider the family $\{ T(M, U) = \{ f \in C(G, G) | f(M) \subset U \} \}$, where $C(G, G) = \{ f \in G^G | f \text{ is continuous} \}$ and $M$ runs over $S$ and $U$ over $\mathcal{U}$. The topology which has $\{ T(M, U) \}$ as a subbase is called the S-topology of $G$. 
For each \( g \in G \), there are two continuous automorphisms \( r_g \) and \( l_g \) of \( G \) which are defined as follows:

\[
\begin{align*}
\forall \ g \in G: \\
\forall \ x \in G: \\
r_g \colon G \to G \\
x \mapsto xg \\
l_g \colon G \to G \\
x \mapsto gx
\end{align*}
\]

Then the map \( \eta_r \colon G \to G^G \) (\( g \to r_g \)) and \( \eta_l \colon G \to G^G \) (\( g \to l_g \)) will be called the right (left) canonical embedding into \( G^G \), where \( G^G \) is a topological group with product topology as before. In this case, \( G \) is homeomorphic with \( \eta_r \) (\( G \)) \( \subset G^G \). Furthermore, \( G \) is homeomorphic to a subset of \( C(G, G) \), is a topological group with relative topology induced from the product topology of \( G^G \). We shall put \( P \) = the product topology of \( G^G \) and \( S \) = the \( S \)-topology on \( C(G, G) \). Then, if \( G \) is a \( T_1 \)-space, we can easily see that the relative topology on \( C(G, G) \) from \( P \) is coarser than the \( S \)-topology on \( C(G, G) \). That is,

\[ P = P \subset S \] on \( G \).

**Proposition 1.1.** If \( S' \) is the family of all closed subsets with nonempty interiors, and if \( S' \) denotes the \( S \)-topology, then following hold.

(i) \( S' \subset S \)

(ii) If \( G \) is a regular space then \( S' \supset P = \emptyset \)

(iii) If \( G \) is a normal space then \( S' \supset S = S \supset P = \emptyset \).

**Proof.** The subbase of the \( S' \)-topology on \( C(G, G) \) is the family \( \{ T(M, U) \} \), where \( M \) runs over \( S' \) and \( U \) runs over \( \emptyset \). Since \( S' \subset S \subset S' \), it is obvious.

Next, let \( G \) be a regular space. Then \( G \) is a \( T_1 \)-space, and thus each point of \( G \) is a closed subset of \( G \). The family \( \{ T(x, U) = \{ r_x \mid x \in G, r_x(x) = x \in U \} \} \) is the subbase of the topology \( P = \emptyset \) on \( G \), where \( U \in \emptyset \). Since \( x \in S \), \( T_x(x, U) \in S \), which means that \( P = \emptyset \). Since \( G \) is regular, for each \( x \in G \) and an open neighborhood \( V \) on \( x \) there exists an open neighborhood \( U(x) \) of \( x \) such that \( U(x) \subset V \), where \( U(x) \) is the closure of \( U(x) \). In this case \( V(x) \in S' \). Since \( x \in V(x) \)

\[
\begin{align*}
T_x(V(x), U) \subset T_x(x, U) \subset U \subset P \circ \text{ on } G \text{ for } T_x(V(x), U) = \{ r_x \mid x \in G, r_x(V(x) \subset U) \}. \\
T_x(V(x), U) \subset T_x(x, U) \subset U \subset P \circ \text{ on } G \text{ for } T_x(V(x), U) = \{ r_x \mid x \in G, r_x(V(x) \subset U) \}. \\
\end{align*}
\]

Since \( x \in V(x) \)
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because of that as in the diagram

\[ G - V(x) \] (a closed subset of \( G \)) and \( x \) are separated by two disjoint open sets, that is, there exists an open subset \( V_1(x) \) such that \( x \in V_1(x) \subset V(x) \), we have

\[ \bigcup_{x \in V(x)} T(x, U) = T(x, U). \]

Therefore \( T(x, U) \) (in \( G \)) is an open subset in the topology \( S_7 \) on \( G \), and thus \( P \subseteq S_7 \). Let \( G \) be a normal space. By (i) and (ii), it suffices to prove that \( S \subseteq S_7 \) for (iii). Let a closed subset \( F \) of \( G \) have no any interior points. As in the diagram

we take an open neighborhood \( V(F) \) of \( F \), then \( G - V(F) \) (a closed subset of \( G \)) and \( F \cap (G - V(F)) = \emptyset \) and \( F \) are separated by two disjoint open subsets. Thus there is an open subset \( V_1(F) \) such that \( F \subseteq V_1(F) \subseteq V(F) \).

In this case \( \overline{V(F)} \) and \( \overline{V_1(F)} \) are in \( S \). Since

\[ \bigcap_{F \subseteq V(F)} V(F) = F \]
we have \( T_r(F, U) = T_r(F, U) \),
where \( T_r(F, U) = \{ r_x \mid x \in G, \ r_x(F) \subset U \} \). Therefore we have \( S_T \subset S'_T \). q.e.d.

\section{2. Dual Groups}

\textbf{Definition 2.1.} Let \( G \) and \( R \) be an abelian (additive) topological group and the topological group consisting of all real numbers, respectively. For the set \( Z \) of all integers \( R/Z \) is the one-dimensional torus. If \( \delta : G \to R/Z \) is a continuous homomorphism, then \( \delta \) is called a \textit{character} of \( G \). The set of all characters of \( G \) is called the dual group of \( G \), written \( G' \).

For \( \sigma_1 \) and \( \sigma_2 \in G \) and \( x \in G \) we define
\[
(\sigma_1 + \sigma_2)(x) = \sigma_1(x) + \sigma_2(x),
\]
then with this addition \( G' \) is an abelian (additive) group. Furthermore, by \( S \)-topology \((\S 1)\) \( G' \) becomes a topological group. (Note that if \( G \) is a Hausdorff space then the \( S \)-topology is the compact open topology i.e., the \( k \)-topology). In particular, if \( G \) is a Hausdorff space then \( G' \) is also a Hausdorff space.

\textbf{Definition 2.2.} A subset \( U \) of a group \( G \) is said to be \textit{symmetric} if \( U = U^{-1} \). In this case \( G \) is an abelian group, \( U \) is symmetric if \( U = -U \).

\textbf{Lemma 2.3.} Let \( e \) be the identity of a topological group \( G \). Then for each open neighborhood \( W \) of \( e \) there exists a symmetric open neighborhood \( U \) of \( e \) such that

\[
U_1 \cdots U_n \subset W,
\]
where \( e_i = \pm 1 \) for \( i = 1, 2, \cdots, n \).

\textit{Proof:} At first, we shall prove the existence of \( U \). Let \( \{ V \} \) be a fundamental system of open neighborhoods of \( e \). For each \( V \in \{ V \} \), since \( G \) is a topological group \( V^{-1} \) is also an open neighborhood of \( e \). Put \( U = V \cap V^{-1} \), then \( U \) is a symmetric open neighborhood of \( e \), since \( U^{-1} = (V \cap V^{-1}) = V^{-1} \cap (V^{-1})^{-1} = V^{-1} \cap V \). Thus each \( V \) contains a symmetric open neighborhood of \( e \). This means that there is a fundamental system \( \{ U \} \) of symmetric neighborhood of \( e \).

In a topological group the mapping \( G \times G \to G((x, y) \to xy^{-1}) \) is continuous. Therefore, for each open subset \( W \) containing \( e \) there exists a symmetric open neighborhood \( U_1 \) such that \( U_1 U_1^{-1} \subset W \). Put \( W_1 = U_1 \cdot U_1^{-1} \), which is an
open subset containing $e$. Then, by the same reason as above there exists a symmetric open neighborhood $U_2$ of $e$ such that $U_2U_2^{-1} \subset W_i$. Repeating this way, we get a symmetric open neighborhood $U$ such that $U^i \cdots U^s \subset W$. q. e. d.

**Lemma 2.4.** $\mathbb{R}$ and $\mathbb{R}/\mathbb{Z}$ are locally homeomorphic.

**Proof.** For each positive integer $m$, $-\frac{1}{m}$ is $\frac{m-1}{m}$ in $\mathbb{R}/\mathbb{Z}$.

As in the diagram

\[
\begin{array}{c}
\text{an open neighborhood } \left( -\frac{1}{m}, \frac{1}{m} \right) \text{ of } 0 \text{ in } \mathbb{R} \text{ is homeomorphic to the open neighborhood } \left( \frac{m-1}{m}, \frac{1}{m} \right) \text{ (see the above diagram). Thus } \mathbb{R} \text{ and } \mathbb{R}/\mathbb{Z} \text{ are locally homeomorphic. q. e. d.}
\end{array}
\]

**Definition 2.5.** (Equicontinuous set) Let $E$ and $F$ be two topological spaces. A family $L$ of functions in $\mathcal{C}(E, F) \subset F^E$ where $F$ is a uniform space, is called an equicontinuous set at $x \in E$ if for each element $V$ in the uniformity of $F$ there exists an open neighborhood $U$ of $x$ such that $(f(y), f(x)) \in V$ for all $y \in U$ and $f \in L$ ([1]~[3]).

The $P$-closure ($P$ = product topology) of an equicontinuous set in $F^E$ is also equicontinuous. On each equicontinuous set in $\mathcal{C}(E, F)$, the relative topology of $k$ coincide with the relative topology of $P$. Above equicontinuous sets there is the Ascoli's theorem ([2]) such that if $E$ is a Hausdorff locally compact topological space and $F$ a Hausdorff uniform space, and a subset $L$ of $\mathcal{C}(E, F)$ with the $k$-topology then $L$ is $k$-compact if and only if
If \( G \) is a locally compact Hausdorff abelian topological group, then its dual group \( G' \) is also a locally compact Hausdorff abelian topological group.

**Proof.** Since the dual group \( G' \) is a Hausdorff abelian topological group, we have to prove that \( G' \) is locally compact. To do this, we take \( U \) which is a neighborhood of \( 0 \) of \( G \) such that \( U \) is compact. Let \( V_m \) be an open neighborhood of \( O \) in \( \mathbb{R}/\mathbb{Z} \) such that \( V_m = \{x \in \mathbb{R}/\mathbb{Z} \mid -\frac{1}{m} < x < \frac{1}{m}, \ m \ \text{is a positive integer}\} \) (see Lemma 2.4.), and let

\[
(R/\mathbb{Z})_m = \{x' \in G' \mid x'(x) \in V_m \ \text{for} \ x \in U\}.
\]

Then \( (R/\mathbb{Z})_m \) is an open neighborhood of \( O' \) of \( G' \) in the \( k \)-topology. We want to prove that \( (R/\mathbb{Z})_m \) is compact in the \( k \)-topology. Using Lemma 2.3, we can prove that there is an open neighborhood \( U_1 \) of \( O \) of \( G \) such that for all \( x' \in (R/\mathbb{Z})_m \) and \( x, y \in U_1, \ x - y \in U_1 \) implies

\[
|x'(v)| = |x'(x) - x'(y)| < \varepsilon.
\]

This means that \( (R/\mathbb{Z})_m \) is equicontinuous (Definition 2.5.). By the above remark, \( (R/\mathbb{Z})_m \) in the \( P \)-topology is also equicontinuous. Therefore \( (R/\mathbb{Z})_m \) in the \( k \)-topology is also equicontinuous. Note that \( R/\mathbb{Z} \) is a uniform space. Since \( R/\mathbb{Z} \) is compact, for each \( x \in G \) and any subset \( H \) of \( G \), \( H(x) = \{f(x) \mid f \in H\} \) is relatively compact in \( R/\mathbb{Z} \). Thus, by the Ascoli's Theorem \( (R/\mathbb{Z})_m \) in the \( k \)-topology is \( k \)-compact, and hence the identity \( O' \) of \( G' \) has a compact \( k \)-open neighborhood \( (R/\mathbb{Z})_m \). Since \( G' \) is a topological group, it is a locally compact space. q.e.d.

§3. Examples

By Theorem 2.6, if \( G \) is a Hausdorff locally compact abelian topological group then so is its dual group \( G' \).

**Theorem 3.1.** Under the above situation

(i) if \( G \) is compact then \( G' \) is discrete.

(ii) if \( G \) is discrete then \( G' \) is compact.
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(Note that a discrete space is locally compact).

**Proof.** For (i) suppose that $G$ is compact. We want to prove that there exists an open neighborhood $V$ of $O$ of $R/Z$ such that $T(G, V) = \{O'\}$. Let us put $V = \{xR/Z | -\frac{1}{m} < x < \frac{1}{m}, \ m \text{ is a positive integer}\}$ (see Lemma 2.4).

Assume that $T(G, V) \neq \{O'\}$, then there is an element $x' \in T(G, V)$ such that for some $x \in G$, $x'(x) \neq 0$. Then by Lemma 2.3, there exists an open neighborhood $V_1$ of $O$ of $R/Z$ such that

$$\sum_{i=1}^{n} V_i \subset V, \ V_i = V_1 \text{ for } i = 1, 2, \ldots, n,$$

and $nx'(x) > \frac{1}{m}$ for a large positive integer $n$. But then we have $nx'(x) = x'(nx) \in V$ (see Lemma 2.3).

This is contrary to the choice of $n$ and $V_1$, and thus $x'(x) = 0$ i.e., $x' = O'$.

For (ii), assume that $G$ is discrete. Then $\{O\}$ is a compact open neighborhood of $O$ of $G$. Thus for each neighborhood $V$ of $O$ of $R/Z$ and for each $x' \in G$, $x'(O) = 0 \in V$. This means that $T(\{O\}, V) = G = \{O\}$ is compact (see theorem 2.6). Thus the proof of (ii) is completed.

**Example 3.2.** The dual group $R$ of $R$ is homeomorphic to $R$.

**Proof.** Recall that $R$ is a one-dimensional vector space over itself. Each linear transformation $f: R \to R$ is a continuous homomorphism. Let $p: R \to R/Z$ be the canonical projection. Then $p$ is a continuous homomorphism. For a linear transformation $f: R \to R$ $pf: R \to R/Z$ is a character of $R$. It is easily proved that each character of $R$ has the form $pf$ for some linear transformation.

Let $M(R)$ be the set of all linear transformations from $R$ to $R$. Then the topological space $M(R)$ with $k$-topology is homeomorphic to $R$.

If for two linear transformations $f \neq g \in M(R)$ then $pf \neq pg$. Hence $R' \cong M(R)$ as topological groups. Hence $R' \cong R$ and $R'$ is also a Hausdorff locally compact abelian topological group.

**Example 3.3.** Let $R$ be a discrete abelian topological group consisting of all real numbers. Then $R'$ is equal to $M(R)$ as sets. But with the $k$-topology given on $R'$ it is compact.

**Example 3.4.** Let $Z$ be a discrete abelian topological group consisting of
all integers. Then \( Z \) is homeomorphic to \( S' \), where \( S' \) is the one-dimensional space.

**Proof.** For each linear transformation \( f \in M(\mathbb{R}) \) \( f/Z \) is a continuous homomorphism from \( Z \) to \( R/Z \). In fact, \( f \) is a some real number \( \sigma \in \mathbb{R} \), and for \( m \in Z \) \( f(m) = \sigma m \mod Z \). For \( \frac{1}{n} \) and \( m + \frac{1}{n} \) and \( e \in Z \)

\[
\frac{e}{n} \equiv (m + \frac{1}{n})e \mod Z.
\]

Therefore we have

\[ Z' \cong S' \cong R/Z \]

as topological groups. Of course \( Z' \) is compact. q.e.d.

**References**


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