A REMARK ON GOLDBACH'S CONJECTURE

By C. J. Mozzochi

The notation here is the same as that found in [2], and this paper might well be considered an appendix to Chapter 3 of that text.

We assume \( n > C_1 \). For each \( n \) let \( E_n \) be those points in \([x_0, x_0 + 1]\) which are not in any closed neighborhood of radius \( x_0 \) about any rational number \( \frac{h}{q} \) where \((h, q) = 1, (q, n) = 1, \) and \( q \leq \log^{15} n \).

It is a trivial consequence ([2] p. 62) of the Prime Number Theorem that

\[
\int_{E_n} f^2(x, n) e(-nx) dx = o(n \log^{-1} n).
\]

In this paper I show that if this estimate could be improved to \( o(n \log^{-\lambda} n) \) for some \( \lambda > 2 \), then it would follow that every sufficiently large even integer can be expressed as the sum of two primes.

The result follows from a suitable modification of the construction found in Chapter 3 of [2]. Without any loss of generality we will assume that \( \lambda \) is arbitrarily close to 2.

Let \( r(n) \) be the number of representations of \( n \) as a sum of two primes. It is easy to see that

\[
r(n) = \int_{x_0}^{x_0 + 1} f^2(x, n) e(-nx) dx \quad \text{for any } x_0.
\]

We decompose the above integral into

\[
r(n) = \int_{E_n} f^2(x, n) e(-nx) dx + \sum_{q \leq \log^4 n} \sum_{0 < h \leq q \atop (q, n) = 1 \atop (h, q) = 1} T(h, q)
\]

(100)

where

\[
T(h, q) = \int_{\frac{h}{q} - x_0}^{\frac{h}{q} + x_0} f^2(x, n) e(-nx) dx
\]

(101)

It follows immediately from Theorem 58 in [2] and the trivial inequalities \(|f(x, n)| \leq n\) and \(|g(y, n)| \leq n\) and the fact that if \(|a| \leq n\) and \(|b| \leq n\), then \(|a^2 - b^2| \leq 2n|a - b|\) with \(a = f\left(\frac{h}{q} + y, n\right)\) and \(b = \frac{\mu(q)}{\phi(q)} g(y, n)\) that if \((h, q) = 1, \)
By a change of variable $y = \left( x - \frac{h}{q} \right)$ we have

$$T(h, q) = \varepsilon \left( -\frac{nh}{q} \right) \int_{-x_0}^{x_0} f^2 \left( \frac{h}{q} + y, n \right) \varepsilon (-ny) dy$$

(103)

However,

$$\left| \varepsilon \left( -\frac{nh}{q} \right) \int_{-x_0}^{x_0} f^2 \left( \frac{h}{q} + y, n \right) \varepsilon (-ny) dy - \frac{\mu^2(q)}{\phi^2(q)} \int_{-x_0}^{x_0} g^2(y, n) \varepsilon (-ny) dy \right|$$

$$\leq \int_{-x_0}^{x_0} f^2 \left( \frac{h}{q} + y, n \right) - \frac{\mu^2(q)}{\phi^2(q)} g^2(y, n) \, dy \leq 2 \int_{-x_0}^{x_0} n^2 \log^{-69} n \, dy$$

$$= 4x_0 n^2 \log^{-69} n = 4n \log^{-54} n.$$ 

Now let

$$T_1(n) = \int_{-x_0}^{x_0} g^2(y, n) \varepsilon (-ny) dy.$$ 

so that by (103) and the above we have that if $(h, q) = 1$ and $q \leq \log^{15} n$

$$\left| T(h, q) - \frac{\mu^2(q)}{\phi^2(q)} T_1(n) \varepsilon \left( -\frac{nh}{q} \right) \right| \leq 4n \log^{-54} n$$

(104)

Let

$$T(n) = \sum_{m_1, n_2} \log^{-1} m_1 \log^{-1} m_2.$$ 

(105)

With the condition of summation $m_1 \geq 2$, $m_2 \geq 2$, and $m_1 + m_2 = n$.

It is easy to see that

$$T(n) = \int_{-\frac{1}{2}}^{\frac{1}{2}} g^2(y, n) \varepsilon (-ny) dy$$

(106)

Also, it is clear that the number of terms on the right-hand side of (105) is $(n-3)$, and each term is greater than $\log^{-2} n$ and less than 1; so that

$$\frac{1}{3} n \log^{-2} n < T(n) < n$$

(107)

Now
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\[ \left| \sum_{m=2}^{m_1} e \left( my \right) \right| \leq \frac{1}{|\sin \pi y|} \leq \frac{1}{2|y|}; \quad (m_1 \geq 2, 0 < |y| \leq \frac{1}{2}). \]

Hence by definition of \( g(y, n) \) and Abel's lemma,

\[ |g(y, n)| < |y|^{-1} \quad (0 < |y| \leq \frac{1}{2}); \]

so that

\[ |T(n) - T_1(n)| \leq 2 \int_{x_1}^{x_2} y^{-2} dy = 2x_0^{-1} = 2n \log^{-15} n \quad (108) \]

Hence, for \((h, q) = 1, q \leq \log^{15} n\)

\[ \left| \epsilon \left( \frac{-nh}{q} \right) \right| \left| \frac{\mu^2(q)}{\phi^2(q)} \right| \left| T(n) - T_1(n) \right| \leq \frac{1}{\phi^2(q)} (2n \log^{-15} n), \]

and combining this fact with (104) we have:

\[ (109) \]

For \((h, q) = 1, q \leq \log^{15} n\)

\[ \left| T(h, q) - \frac{\mu^2(q)}{\phi^2(q)} T(n) \epsilon \left( \frac{-nh}{q} \right) \right| \leq 4n \log^{-54} n + \frac{1}{\phi^2(q)} (2n \log^{-15} n) \]

so that adding (109) \( \phi(q) \) times for some fixed \( q \leq \log^{15} n \) we have:

\[ (110) \]

But \( \phi(q) \leq \log^{15} n \) and \( \left( \lfloor 4 \right) \) p. 55

\[ \sum_{0 < h \leq q} \epsilon \left( \frac{-nh}{q} \right) = C_q(n); \]

so that it follows immediately from (110) that:

\[ (111) \]

Considering only those \( q \leq \log^{15} n \) such that \((q, n) = 1\) we have:

\[ (112) \]
\[ \leq 4n \log^{-24}n + C_1(2n \log^{-10}n) \leq C_2 n \log^{-10}n; \]
since by Theorem 327 in [4]
\[ \sum_{q \leq \log^{10}n} \frac{1}{\phi^{4/3}(q)} \leq C_1 \ (C_1 \text{ independent of } n). \]
Hence combining (100), (112) and the unproved statement:
\[ \left| \int_{E} f^2(x, n) \ v (-nx) \ dx \right| \leq C_3 n \log^{-d}n \text{ for some } d > 2 \]
we have
\[ \left| \sum_{q \leq \log^{10}n} \frac{\mu^2(q)}{\phi^{2}(q)} C_q(n) \right| \leq C_4 n \log^{-d}n \]
(113)
Let
\[ S(n) = \sum_{q=1}^{\infty} \frac{\mu^2(q)}{\phi^{2}(q)} C_q(n) D_q(n) \]
where
\[ D_q(n) = \begin{cases} 1 & \text{if } (q, n) = 1 \\ 0 & \text{if } (q, n) > 1 \end{cases} \]
\[ \left| S(n) - \sum_{q \leq \log^{10}n} \frac{\mu^2(q)}{\phi^{2}(q)} C_q(n) \right| \leq \sum_{q > \log^{10}n} \frac{\mu^2(q)}{\phi^{2}(q)} C_q(n) \]
\[ \leq \sum_{q > \log^{10}n} \frac{1}{\phi^{2}(q)} \]
since \( \mu^2(q) = 0 \) if \( q \) is not square free, and by Theorem 272 in [4] if \( q \) is square free and \( (q, n) = 1 \), then \( |C_q(n)| = 1 \). Hence
\[ \left| S(n) - \sum_{q \leq \log^{10}n} \frac{\mu^2(q)}{\phi^{2}(q)} C_q(n) \right| \leq C_5 \log^{-14}n, \]
by Theorem 327 in [4].
Combining this fact with (107) and (113) we have
\[ \left| S(n)T(n) - T(n) \sum_{q \leq \log^{10}n} \frac{\mu^2(q)}{\phi^{2}(q)} C_q(n) \right| \leq C_5 n \log^{-14}n. \]
(114)
Combining (113) and (114) we have
\[ |r(n) - S(n)T(n)| \leq C_6 n \log^{-d}n \]
(115)
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Let
\[ f(q) = \frac{\mu^2(q)}{\phi^2(q)} - C_q(n)D_q(n) \]

By Theorem 60, Theorem 67, and Theorem 262 in [4], \( f \) is a multiplicative function of \( q \). Also,
\[ \sum_{q=1}^{\infty} |f(q)| \leq \sum_{q=1}^{\infty} \frac{1}{\phi^2(q)} < \infty \]
so that by Theorem 2 in [2] we have for each \( n \):
\[ S(n) = \prod_{p \mid m=0} \sum_{p} f(p^m). \]

But
- If \( m=0 \), \( f(p^m) = f(p^0) = f(1) = \frac{\mu^2(1)}{\phi^2(1)} C_1(n)D_1(n) = 1. \)
- If \( m=1 \), \( f(p^1) = f(p) = \frac{\mu^2(p)}{\phi^2(p)} C_p(n)D_p(n) = \frac{C_p(n)D_p(n)}{(p-1)^2}; \)
- If \( m \geq 2 \), \( \mu(p^m) = 0 \); so that \( f(p^m) = 0 \); so that
\[ S(n) = \prod_{p \mid m \geq 2} \left( 1 + \frac{C_p(n)D_p(n)}{(p-1)^2} \right). \]

Clearly, if \( n \) is even, \( D_2(n) = 0 \); so that since \( C_p(n) = (p-1) \) if \( (p, n) > 1 \) and \( C_p(n) = -1 \) if \( (p, n) = 1 \).
\[ S(n) = \prod_{p > 2} \left( 1 + \frac{C_p(n)D_p(n)}{(p-1)^2} \right) \geq \prod_{p > 2} \left( 1 - \frac{1}{(p-1)^2} \right) \]
\[ \geq \prod_{m=2}^{\infty} \left( 1 - \frac{1}{m^2} \right) = \frac{1}{2}. \]

Combining this fact with (107) and (115) it follows that every sufficiently large even integer can be expressed as the sum of two primes.

REMARK. Let \( x_0^* = x_0^{n^{-e}} \) where \( 0 < e < 1 \). Let \( E_n^* \) be those points in \([x_0^*, \infty)\) which are not in any closed neighborhood of radius \( x_0^* \) about any rational number \( \frac{h}{q} \) where \( (h, q) = 1 \) and \( q \leq \log^{15} n \). Clearly
\[ E_n \subset (E_n^* \cup E_n^{**}) \]
where
\[ E_n^{**} = \bigcup_{(h, q) = 1} \left[ \frac{h}{q} - x_0^*, \frac{h}{q} + x_0^* \right]. \]
But

\[ E_n^* \cap E_n^{**} = \emptyset \]

and

\[ \int_{E_n^{**}} \left| f^2(x, n) \right| dx \leq 2x_0 n^2 \log^{30} n \leq C_f(n \log^{-3} n); \]

so that if

\[ \int_{E_n^{**}} \left| f^2(x, n) \right| dx = o\left( \frac{n}{\log^A n} \right) \text{ for some } A > 2, \]

then

\[ \int_{E_n^{**}} f^2(x, n) e(-nx) dx = o\left( \frac{n}{\log^A n} \right) \text{ for some } A > 2. \]

Theorem 56 on page 54 in [2] states that if (153): \( n \log^{-3} n < v \leq n, \)

(154): \( \log^{15} n < q \leq n \log^{-15} n, \)

(155): \( (h, q) = 1, \) then

(156): \( \left| f\left( \frac{h}{q}, v \right) \right| = o\left( n \log^{-3} n \right). \)

Fix \( \epsilon > 0, \) arbitrarily small. Consider

(153)*: \( n^{1/2} \log^{-(1+\epsilon)} n < v \leq n; \)

(154)*: \( \log^{15} n < q \leq n^{1+\epsilon} \log^{-15} n; \)

(156)*: \( \left| f\left( \frac{h}{q}, v \right) \right| = o\left( n^{1/2} \log^{-(1+\epsilon)} n \right). \)

It is easy to see (cf. [2] p. 62) that if it could be shown that (153)*, (154)* and (155) imply (156)*, then (116) would follow.

All my results may be known to others.

Box 1315
Hartford, Conn. 06101
U. S. A.

REFERENCES


