INFINITESIMAL VARIATIONS OF ANTI-INVARIANT SUBMANIFOLDS OF A KAEBLERIAN MANIFOLD

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Introduction

Various authors (see, for example, [1], [7], [8], [9]) studied recently anti-invariant (or totally real) submanifolds of a Kaehlerian manifold.

On the other hand, one of the present authors [6] has studied infinitesimal variations of submanifolds applying the method developed in [3] and [4].

The main purpose of the present paper is to study infinitesimal variations which carry an anti-invariant submanifold into an anti-invariant submanifold. Such an infinitesimal variation will be called in this paper an anti-invariant variation.

In §1, we state formulas for anti-invariant submanifolds of a Kaehlerian manifold which we need later.

§2 is devoted to the study of infinitesimal variations which carry an anti-invariant submanifold into an anti-invariant submanifold. A necessary and sufficient condition for an infinitesimal variation to carry an anti-invariant submanifold into an anti-invariant submanifold is given by Theorem 2.1.

In §3, we consider what we call infinitesimal parallel variations and prove that a parallel variation is an anti-invariant variation.

In §4 and 5, we compute variations of $f_0^x$ and $f_3^x$ respectively and in §6, we study isometric variations.

The last §7 is devoted to the study of variations of the second fundamental tensors. In the later part of §7, we study anti-invariant normal variations which preserve $f_0^x$ and mean curvature vector.

1. Anti-invariant submanifolds of a Kaehlerian manifold

Let $M^{2m}$ be a real $2m$-dimensional Kaehlerian manifold covered by a system of coordinate neighborhoods $(U; x^i)$ and with almost complex structure tensor $F_i^k$ and
Hermitian metric tensor \( g_{ji} \), where, here and in the sequel, the indices \( h, i, j, k, \ldots \) run over the range \( \{1, 2, \ldots, 2m\} \). Then we have

\[
F_i^l F^h_l = -\delta_i^h, \quad F_i^l F^s_l g_{ts} = g_{ji}
\]

(1.1)

where \( \nabla_j \) denotes the operator of covariant differentiation with respect to the Christoffel symbols \( \Gamma^h_{ji} \) formed with \( g_{ji} \).

Let \( M^n \) be an \( n \)-dimensional Riemannian manifold covered by a system of coordinate neighborhoods \( \{V_y^a\} \) and with metric tensor \( g_{cb} \), where, here and in the sequel, the indices \( a, b, c, \ldots \) run over the range \( \{1, 2, \ldots, n\} \). We assume that \( M^n \) is isometrically immersed in \( M^{2m} \) by the immersion \( i: M^n \rightarrow M^{2m} \) and identify \( i(M^n) \) with \( M^n \). We represent the immersion \( i: M^n \rightarrow M^{2m} \) locally by

\[
x^h = x^h(y^a)
\]

(1.3)

and put

\[
B^h_b = \partial_b x^h, \quad (\partial_b = \partial/\partial y^b),
\]

(1.4)

which are \( n \) linearly independent vectors of \( M^{2m} \) tangent to \( M^n \).

Since the immersion \( i \) is isometric, we have

\[
g_{cb} = g_{ji} B^j_c B^i_b
\]

(1.5)

We denote by \( C^h_x \) \( 2m-n \) mutually orthogonal unit normals to \( M^n \), where, here and in the sequel, the indices \( x, y, z, \ldots \) run over the range \( \{n+1, n+2, \ldots, 2m\} \). Then the equations of Gauss are written as

\[
\nabla_c B^h_b = h^x_{cb} C^h_x,
\]

(1.6)

where \( \nabla_c \) denotes the operator of van der Waerden-Bortolotti covariant differentiation along \( M^n \) and \( h^x_{cb} \) are second fundamental tensors of \( M^n \) with respect to the normals \( C^h_x \) and those of Weingarten as

\[
\nabla_c C^h_x = -h^x_{ca} B^h_a,
\]

(1.7)

where

\[
h^x_{ca} = h^x_{cb} g^{ba} = h^x_{cb} g^{ba} g_{sa},
\]

\( g^{ba} \) denoting covariant components of the metric tensor \( g_{cb} \) of \( M^n \), and \( g_{sa} \) the metric tensor of the normal bundle.

If the transform by \( F \) of any vector tangent to \( M^n \) is always normal to \( M^n \),
that is, if there exists a tensor field \( f_b^x \) of mixed type such that

(1.8) \[ F^i_j B^i_b = -f_b^x C^h_x, \]

we say that \( M^n \) is \textit{anti-invariant} (or \textit{totally real}) in \( M^{2m} \).

For the transform by \( F \) of normal vectors \( C_x^h \), we have equations of the form

(1.9) \[ F^i_j C^i_y = f^a_j B^h_a + f^x_y C^h_x, \]

where

(1.10) \[ f^a_y = f^x_b g^{ba}_{zy}, \]

which can also be written as

(1.11) \[ f_{yx} = f_{ay}, \]

where \( f_{yx} = f^b_j g^{ba}_{zy} \) and \( f_{ay} = f^x_a g^{ax}_{zy} \).

From (1.8) and (1.9) we find (cf. [7], [9])

(1.12) \[ f^y_j f^a_j = \delta^a_j, \]

(1.13) \[ f^y_j f^x_j = 0, \]

(1.14) \[ f^x_j f^a_j = 0, \]

(1.15) \[ f^x_j f^x_j = -\delta^x_j + f^a_j f^x_a. \]

Equations (1.14) and (1.15) show that \( f^x_y \) is an \( f \)-structure in the normal bundle of \( M^n \) if it does not vanish. Differentiating (1.8) and (1.9) covariantly along \( M^n \), and using equations of Gauss and Weingarten, we find

(1.16) \[ h^{x_j f^a_j}_b - h^a_j f^x_j_b = 0, \]

(1.17) \[ \nabla f^x_j = -h^{x_j f^y}_j, \]

(1.18) \[ \nabla f^a_j = h^a_j f^x_j, \]

(1.19) \[ \nabla f^x_j = h^a_j f^x_j - h^a_j f^x_a. \]

If \( m = n \), from (1.12) we have \( f^a_j f^x_j = \delta^x_j \), and consequently from (1.15) we find

\[ f^x_j f^x_j = 0, \]

that is, \( f_{xy} f^{xy} = 0 \), \( f_{xy} = f^x_j g_{xy} \) and \( f^{xy} = f^x_j g^{xj} \) being skew-symmetric.

Thus we have \( f^x_j = 0 \). In this case, equations (1.12)~(1.15) reduces to

(1.20) \[ f^a_j f^a_j = \delta^a_j, \quad f^x_j f^x_j = \delta^x_j, \]
2. Infinitesimal variations of anti-invariant submanifolds

We consider an infinitesimal variation of anti-invariant submanifold $M^n$ of a Kaehlerian manifold $M^{2m}$ given by

\[(2.1) \quad x^h = x^h(y) + \zeta^h(y)e,\]

where $\zeta^h(y)$ is a vector field of $M^{2m}$ defined along $M^n$ and $e$ is an infinitesimal. We then have

\[(2.2) \quad \overline{B}^h_b = B^h_b + (\partial \xi^h_b)e,\]

where $\overline{B}^h_b = \delta^h_b x^h$ are $n$ linearly independent vectors tangent to the varied submanifold. We displace $\overline{B}^h_b$ parallelly from the varied point $(x^h)$ to the original point $(x)$. We then obtain the vectors

\[(2.3) \quad \overline{B}^h_b = B^h_b + \Gamma^h_{ji} B^j_b \xi^i e,\]

neglecting the terms of order higher than one with respect to $e$, where

\[(2.4) \quad \nabla^h_b \xi = \partial_b \xi^h + \Gamma^h_{ji} B^j_b \xi^i e.\]

In the sequel we always neglect terms of order higher than one with respect to $e$. Thus putting

\[(2.5) \quad \delta B^h_b = \overline{B}^h_b - B^h_b,\]

we have from (2.3)

\[(2.6) \quad \delta B^h_b = (\nabla^h_b e).\]

Putting

\[(2.7) \quad \xi^h = \xi^a B^h_a + \xi^a C^h_a,\]

we have

\[(2.8) \quad \nabla^h_b \xi^h = (\nabla^h_b \xi^a - h^a_b \xi^2 e) B^h_a + (\nabla^h_b \xi^2 + h^a_b \xi^a) C^h_a.\]

Now we denote by $\overline{C}^h_y$ $2m-n$ mutually orthogonal unit normals to the varied submanifold and by $\tilde{C}^h_y$ the vectors obtained from $\overline{C}^h_y$ by parallel displacement of $\overline{C}^h_y$ from the point $(x^h)$ to $(x)$. Then we have

\[(2.9) \quad \tilde{C}^h_y = \overline{C}^h_y + \Gamma^h_{ji} (x + \xi e) \xi^i \tilde{C}^h_y e.\]
We put
\[(2.10)\]
\[\partial C^h_y = \dot{C}^h_y - C^h_y\]
and assume that \(\partial C^h_y\) is of the form
\[(2.11)\]
\[\partial C^h_y = \eta^h_y e^{\varepsilon} = (\eta^a_y B^h_e + \eta^x_y C^h_x)\varepsilon.\]

Then, from (2.9), (2.10) and (2.11), we have
\[(2.12)\]
\[\bar{C}^h_y = C^h_y - \Gamma^h_{ji} \xi^i_j C^h_x + (\eta^a_y B^h_e + \eta^x_y C^h_x)\varepsilon.\]

Applying the operator \(\partial\) to \(B^i_j C^h_x g_{ji} = 0\) and using (2.6), (2.8), (2.11) and \(\partial g_{ji} = 0\), we find
\[(2.13)\]
\[\eta^a_y = - (\nabla^a_s + h^a_{ly} \xi^a_x)\varepsilon,\]
\(\nabla^a\) being defined to be \(\nabla^a = g^{ac} \nabla_c\). Applying the operator \(\partial\) to \(C^i_j C^h_x g_{ji} = \partial x^i\) and using (2.11) and \(\partial g_{ji} = 0\), we find
\[(2.14)\]
\[\eta^x_y + \eta^y_x = 0,\]
where \(\eta^x_y = \eta^y_x g_{xy}\).

We now assume that the infinitesimal variation (2.1) carries an anti-invariant submanifold into an anti-invariant submanifold, that is,
\[(2.15)\]
\[F^h_i (x + \xi^i \varepsilon) \bar{B}^i_j \text{ are linear combinations of } \bar{C}^h_x.\]

Now using \(\nabla_j F^h_i = 0\) and (1.8), we see that
\[F^h_i (x + \xi^i \varepsilon) \bar{B}^i_j \]
\[= (F^h_i + \xi^i \partial_j F^h_i \varepsilon) (B^i_j + \partial_b \xi^i \varepsilon)\]
\[= [F^h_i - \xi^i (\Gamma^h_{ji} F^h_i - \Gamma^h_{ji} F^h_i) \varepsilon] (B^i_j + \partial_b \xi^i \varepsilon)\]
\[= F^h_i B^i_j + (F^h_i \nabla_b \xi^i + f^b_j \Gamma^h_{ji} C^h_x \xi^i) \varepsilon,\]
that is, by (2.12),
\[(2.16)\]
\[F^h_i (x + \xi^i \varepsilon) \bar{B}^i_j \]
\[= - f^b_j C^h_x + [F^h_i \nabla_b \xi^i + f^b_j (\eta^a_y B^h_e + \eta^x_y C^h_x)] \varepsilon.\]
Thus we see that (2.15) is equivalent to

(2.17)  \[ F_i \nabla^i B^h_{\alpha \beta} + f^i \eta^a_i B^h_{a} \]  
are linear combinations of \( C_x^h \).

On the other hand, using (2.8) and (2.13), we have

(2.18)  \[ F_i \nabla^i + f^i \eta^a_i B^h_{a} \]

\[ = -(\nabla^{[a} - h^{a}_{b} \xi^{c]} \xi^{h} \xi^{b} f^a \xi^{b} + (\nabla^{[a} + h^{a}_{b} \xi^{c]} \nabla^{b} B^h_{a} + f^a \xi^{b} C_x^h) \]

\[ = [(-\nabla^{[a} \xi^{h} f^a \xi^{b} f^b - f^a \xi^{h} \nabla^{[a} \xi^{c]} \nabla^{b} B^h_{a} + f^a \xi^{b} C_x^h) \]

\[ + [(-\nabla^{[a} f^a \xi^{b} f^b - (\nabla^{[a} f^a \xi^{c]} \nabla^{b} B^h_{a} + f^a \xi^{b} C_x^h) C_x^h. \]

Thus (2.15) or (2.16) is equivalent to

(2.19)  \[ (\nabla^{[a} \xi^{b} f^a \xi^{b} f^b = f^a \nabla^{[a} \xi^{b} f^b + h^{a}_{b} \xi^{c]} \nabla^{b} B^h_{a} + f^a \xi^{b} C_x^h, \]

or, by (1.16), to

(2.20)  \[ (\nabla^{[a} \xi^{b} f^a \xi^{b} f^b = f^a \nabla^{[a} \xi^{b} f^b + h^{a}_{b} \xi^{c]} \nabla^{b} B^h_{a} + f^a \xi^{b} C_x^h, \]

or, by (1.11), to

(2.21)  \[ (\nabla^{[a} \xi^{b} f^a \xi^{b} f^b = (\nabla^{[a} \xi^{b} f^b + h^{a}_{b} \xi^{c]} \nabla^{b} B^h_{a} + f^a \xi^{b} C_x^h. \]

Thus we have

**THEOREM 2.1.** *In order for an infinitesimal variation to carry an anti-invariant submanifold into an anti-invariant submanifold, it is necessary and sufficient that the variation vector \( \xi^h \) satisfies (2.20) or (2.21).*

**COROLLARY 2.1.** *If a vector field \( \xi^h \) defines an infinitesimal variation which carries an anti-invariant submanifold into an anti-invariant submanifold, then another vector field \( \xi^h \) which has the same normal part as \( \xi^h \) has the same property.*

An infinitesimal variation given by (2.1) is called an *anti-invariant variation* if it carries an anti-invariant submanifold into an anti-invariant submanifold. For an infinitesimal variation given by (2.1), when \( \xi = 0 \), that is, when the variation vector \( \xi^h \) is tangent to the submanifold we say that the variation is *tangential* and when \( \xi = 0 \), that is, when the variation vector \( \xi^h \) is normal to the submanifold we say that the variation is *normal.*

Since \( \nabla_c f^a_b \) is symmetric in \( c \) and \( b \) by (1.17), we see that (2.21) is equivalent to
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(2.22) \[ \nabla_b(\xi_x f_a^x) = \nabla_d(\xi_x f_b^x). \]

Thus we see

**Proposition 2.1.** If \( \xi_x f_a^x \) is closed, then an infinitesimal variation is an anti-invariant variation.

If \( m > n \), then there exists a normal vector field \( \xi \) in the normal bundle such that \( \xi_x f_a^x = 0 \). Therefore, from Proposition 2.1, we obtain

**Theorem 2.2.** If \( m > n \), then there always exists an anti-invariant normal variation.

The mean curvature vector \( H^h \) of \( M^n \) is given by \( H^h = \frac{1}{n} g^{ch} \nabla_c B_b^h \). If \( C^h \) is a unit normal vector in the direction of \( H^h \), then \( H^h = \alpha C^h \) for some function \( \alpha \).

We call \( \alpha \) the mean curvature of \( M^n \). If the second fundamental tensors of \( M^n \) is of the form \( h_{ba}^x = g_{ba}^y H^x \), where \( H^x = \frac{1}{n} g^{bd} h_{bd}^x \), then \( M^n \) is said to be totally umbilical.

Now we assume that \( M^n \) is totally umbilical and anti-invariant in \( M^{2m} \), then (1.16) gives

(2.23) \[ H^x f_a^x = 0. \]

From (2.23) and Proposition 2.1, we have

**Theorem 2.3.** Let \( M^n \) be a not totally geodesic, totally umbilical, anti-invariant submanifold of a Kaehlerian manifold \( M^{2m} (m > n) \). Then the normal variation defined by the mean curvature vector \( H^h \) carries \( M^n \) into an anti-invariant submanifold.

If a tangent vector \( u^a \) satisfies

(2.24) \[ \nabla_b u^a = \nabla_d u^d \]

then an infinitesimal normal variation defined by \( \xi^x = f_a^x u^a \) satisfies (2.22).

Therefore we have

**Proposition 2.2.** If a tangent vector \( u^a \) satisfies (2.24), then the normal variation defined by \( \xi^x = f_a^x u^a \) is anti-invariant.

3. Parallel variation

Suppose that an infinitesimal variation \( x^h = x^h + \xi^h e \) carries a submanifold \( x^h = \)
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$x^b(y)$ into another submanifold $\tilde{x}^b = x^b(y)$ and the tangent space of the original submanifold at $(x^b)$ and that of the varied submanifold at the corresponding point $(\tilde{x}^b)$ are parallel. Then we say that the variation is parallel.

Since we have from (2.5), (2.6) and (2.8)

\[ \tilde{B}_b^h = [\delta^a_b + (\nabla_b^a - h^a_b \xi^b)\xi]B^h_a + (\nabla_b^e \xi^e + h_{ba} \xi^e)C_{-}^{bh} \xi, \]

we have

**Lemma 3.1** ([6]). In order for an infinitesimal variation to be parallel, it is necessary and sufficient that

\[ \nabla_b^e \xi^e + h_{ba} \xi^e = 0. \tag{3.2} \]

If (3.2) is satisfied, then (2.19) is satisfied. Thus we have

**Theorem 3.1.** A parallel variation is an anti-invariant variation.

4. Variation of $f^x_b$

Suppose that an anti-invariant variation $\tilde{x}^b = x^b + \xi^b \xi$ carries an anti-invariant submanifold into an anti-invariant submanifold, that is, it is an anti-invariant variation. Then putting

\[ F^h_i(x + \xi \xi)B^i_b = -(f^x_b + \delta f^x_b)C^h_x, \tag{4.1} \]

we have, from (2.16), (2.18) and (2.19),

\[ -(\delta f^x_b)C^h_x = [(\nabla_b^e \xi^e + h_{ba} \xi^e) f^x_y - (\nabla_b^e \xi^e + h_{ba} \xi^e) f^x_y + f^y_{x^x}]C^{h}^{e} \xi, \]

from which

\[ \delta f^x_b = [(\nabla_b^e \xi^e - h_{ba} \xi^e) f^x_a - (\nabla_b^e \xi^e + h_{ba} \xi^e) f^x_y - f^y_{y^x}]\xi. \tag{4.2} \]

Thus we have

**Proposition 4.1.** Suppose that an infinitesimal variation is anti-invariant. Then the variation of $f^x_b$ is given by (4.2).

**Proposition 4.2.** An anti-invariant variation preserves $f^x_b$ if and only if

\[ (\nabla_b^e \xi^e - h_{ba} \xi^e) f^x_a - (\nabla_b^e \xi^e + h_{ba} \xi^e) f^x_y + f^y_{y^x} = 0. \tag{4.3} \]
5. Variation of $f_y^x$

In this section we suppose that an infinitesimal variation $\tilde{x}^h = x^h + \xi^h \varepsilon$ is anti-invariant. To find the variation of $f_y^x$, we apply the operator $\delta$ to

$$F_i^{h C_t} = f_y^h B_i^h + f_x^h C_x^h.$$

Then using $\delta F_i^h = 0$, (2.11) and (2.6), we find

$$F_i^h (\eta^a B_i^h + \eta^x C_x^h) \varepsilon
= (\delta f_y^h) B_i^h + f_y^h (\nabla_\xi^h \xi^h) + (\delta f_x^h) C_x^h + f_x^h (\eta^x B_i^h + \eta^x C_x^h) \varepsilon,$$

or

$$[- \eta_y f_y^x C_x^h + \eta_y (f_y^h B_i^h + f_x^h C_x^h)] \varepsilon
= (\delta f_y^h) B_i^h + f_y^h (\nabla_\xi^h \xi^h - h^a \xi^h) + (\delta f_x^h) C_x^h + f_x^h (\eta^x B_i^h + \eta^x C_x^h) \varepsilon,$$

from which

$$\eta_y f_y^x \varepsilon = \delta f_y^h + f_y^h (\nabla_\xi^h \xi^h - h_y \xi^h) + f_x^h \eta^x \varepsilon - f_y^h (\nabla_\xi^h \xi^h + h_b \xi^h) \varepsilon,$$

or, using (1.18)

(5.1) \hspace{1cm} \delta f_y^h = [\xi \nabla_\xi f_y^h - f_y^h (\nabla_\xi \xi^h - h \xi^h) + f_x^h (\eta^x B_i^h + \eta^x C_x^h)] \varepsilon

and

$$[- \eta_y f_y^x + \eta_y \xi^h \xi^h] \varepsilon = f_y^h (\nabla_\xi^h \xi^h - h_y \xi^h) + \delta f_x^x + f_x^h \eta_x \xi^h,$$

$$\delta f_x^x = [- f_y^h (\nabla_\xi^h \xi^h + h_y \xi^h) + (\nabla_\xi^h \xi^h + h_y \xi^h) f_x^x + \eta_x \eta_x f_x^x] \varepsilon,$$

or, using (1.19),

(5.2) \hspace{1cm} \delta f_x^x = [\xi \nabla_\xi f_x^x + \eta_x f_x^x - f_x^h (\nabla_\xi^h \xi^h) + (\nabla_\xi^h \xi^h) f_x^x] \varepsilon,$$

or, using (2.13),

(5.3) \hspace{1cm} \delta f_y^x = \eta^x f_y^x - \eta_y f_y^x + \eta_y f_y^x - f_y^h \eta_x \xi^h \varepsilon.$$

Thus we have
PROPOSITION 5.1. Suppose that an infinitesimal variation is anti-invariant. Then the variation of $f^x_y$ is given by (5.2) or (5.3).

PROPOSITION 5.2. An anti-invariant variation preserves the $f$-structure $f^x_y$ in the normal bundle if and only if

\[(5.4)\quad \xi^c\nabla_c f^x_y + \eta^x_y f^x + \eta^x_z f^z - f^x_y (\nabla^x_c) f^x_z = 0,\]

or

\[(5.5)\quad \eta^x_c f^x_c + \eta^x_y f^x - \eta^x_z f^z - f^x_y \eta^x_z = 0.\]

6. Isometric variations

First of all, applying the operator $\delta$ to (1.5) and using (2.6), (2.8) and $\delta g_{ji} = 0$, we find (cf. [6])

\[(6.1)\quad \delta g_{cb} = (\nabla^x_c + \nabla^x_b - 2h_{cb} \xi^x) \xi_c,\]

from which

\[(6.2)\quad \delta g^{ba} = -(\nabla^a_b + \nabla^b_a - 2h^{ba} \xi^a) \xi_c.\]

A variation of a submanifold for which $\delta g_{cb} = 0$ is said to be isometric.

Now we assume that an anti-invariant variation preserves $f^x_b$, that is, $\delta f^x_b = 0$. Then (1.12), (1.14) and (4.3) imply

\[(6.3)\quad \nabla^a_y - h_{ba} \xi^a = f^y_b f^x_x.\]

Thus, by (2.14), (6.1) and (6.3), we have $\delta g_{cb} = 0$. Therefore we obtain

PROPOSITION 6.1. If an anti-invariant variation preserves $f^x_b$, then the variation is isometric.

We assume next that $m = n$ and the anti-invariant variation is normal. Then we have $f^x_y = 0$ and hence (4.2) becomes

\[(6.4)\quad \delta f^x_y = -(h^a_{ba} f^x_a - f^x_y \eta^x) \xi_c.\]

If the variation moreover preserves $f^x_y$, then (6.1) and Proposition 6.1 show that $h_{cbx} \xi^x = 0$. Thus (6.4) implies $f^x_y \eta^x = 0$, from which $\eta^x = 0$. Consequently
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(2.11) reduces to

\[ \partial C^h = \eta^a \beta^h_a \varepsilon. \]

**Proposition 6.2.** If \( m = n \) and anti-invariant normal variation preserves \( f^x_b \), then the variation of \( C^h_y \) is given by (6.5).

Furthermore, if the variation is parallel, then (2.13) gives \( \eta^a = 0 \). Thus we have

**Proposition 6.3.** If \( m = n \) and if a parallel anti-invariant normal variation preserves \( f^x_b \), then it preserves \( C^h_y \).

7. Variations of the second fundamental tensors

In this section we compute infinitesimal variations of the second fundamental tensors (see [6]).

Suppose that \( \psi^h \) is a vector field of \( \mathcal{M}_{2m} \) defined intrinsically along the submanifold \( \mathcal{M}^n \). When we displace the submanifold \( \mathcal{M}^n \) by \( x^h = x^h + \xi^h(y) \varepsilon \) in the direction of \( \xi^h \), we obtain a vector field \( \psi^h \) which is defined also intrinsically by the same rule along the varied submanifold. If we displace \( \psi^h \) back parallelly from the point \( \langle x^h \rangle \) to \( \langle x^h \rangle \), we get

\[ \psi^h = \psi^h + \Gamma^h_{ji} (x + \xi^h(y) \varepsilon) \xi^i \psi^i \varepsilon \]

and hence, putting \( \partial \psi^h = \psi^h - \psi^h \), we find

\[ \partial \psi^h = \psi^h - \psi^h + \Gamma^h_{ji} \xi^i \psi^i \varepsilon. \]

Similarly we have

\[ \partial \nabla \psi^h = \nabla \psi^h - \nabla \psi^h + \Gamma^h_{ji} \xi^i \nabla \psi^i \varepsilon, \]

that is,

\[ \partial \nabla \psi^h = \nabla \psi^h - \nabla \psi^h + (\partial \Gamma^h_{ji} + \Gamma^h_{kt} \Gamma^t_{ji} \xi^k \beta^i \varepsilon \]

\[ + \Gamma^h_{ji} [(\partial \xi^i) \psi^j + \xi^i (\partial \psi^j)] \varepsilon. \]

On the other hand, we have

\[ \nabla \partial \psi^h = \nabla \psi^h - \nabla \psi^h + (\partial \Gamma^h_{kt} + \Gamma^h_{tj} \Gamma^t_{kt} \xi^k \beta^i \varepsilon \]

\[ + \Gamma^h_{ji} [(\partial \xi^i) \psi^j + \xi^i (\partial \psi^j)] \varepsilon. \]

From these equations we find
\[
\delta \nabla^h_{\partial^i} - \nabla^h_{\partial^i} = K^h_{\partial^i \partial^j} B^j_{\partial^k} e_k,
\]
where \( K^h_{\partial^i \partial^j} \) is the curvature tensor of \( M^{2m} \).

Similarly for a tensor field carrying three kinds of indices, say \( T^h_{\partial^i} \), we have

\[
\delta \nabla^h_{\partial^i} T^h_{\partial^i} - \nabla^h_{\partial^i} \delta T^h_{\partial^i} = K^h_{\partial^i \partial^j} B^j_{\partial^k} T^h_{\partial^k} e - \langle \delta T^a_{\partial^i} \rangle T^h_{\partial^i} e - \langle \delta T^x_{\partial^i} \rangle T^h_{\partial^i} e,
\]

\( \delta T^a_{\partial^i} \) and \( \delta T^x_{\partial^i} \) being the variation of the affine connection \( T^a_{\partial^i} \) induced on \( M^n \) and that of the affine connection induced on the normal bundle of \( M^n \) respectively. Applying formula (7.1) to \( B^h_{\partial^i} \), we find

\[
\delta \nabla^h_{\partial^i} B^h_{\partial^i} - \nabla^h_{\partial^i} \delta B^h_{\partial^i} = K^h_{\partial^i \partial^j} B^j_{\partial^k} B^k_{\partial^l} e - \langle \delta T^a_{\partial^i} \rangle B^h_{\partial^i} e,
\]
or using (1.6) and (2.6)

\[
\delta (h_{\partial^i} C^h_{\partial^i}) = (\nabla^h_{\partial^i} + K^h_{\partial^i \partial^j} B^j_{\partial^k} B^k_{\partial^l}) e - \langle \delta T^a_{\partial^i} \rangle B^h_{\partial^i} e,
\]
from which, using (2.11),

\[
(\delta h_{\partial^i} C^h_{\partial^i}) = (\nabla^h_{\partial^i} + K^h_{\partial^i \partial^j} B^j_{\partial^k} B^k_{\partial^l}) e - \langle \delta T^a_{\partial^i} \rangle B^h_{\partial^i} e,
\]
Thus we have

\[
\delta T^a_{\partial^i} = (\nabla^h_{\partial^i} + K^h_{\partial^i \partial^j} B^j_{\partial^k} B^k_{\partial^l}) B^a_{\partial^i} e - h_{\partial^i} x^a e,
\]
and

\[
\delta h_{\partial^i} x = -h_{\partial^i} x^a e + (\nabla^h_{\partial^i} + K^h_{\partial^i \partial^j} B^j_{\partial^k} B^k_{\partial^l}) C^x_{\partial^i} e,
\]
from which

\[
\delta h_{\partial^i} x = (\nabla^h_{\partial^i} + h_{\partial^i} x^a e + h_{\partial^i} x^a e + h_{\partial^i} x^a e - h_{\partial^i} x^a e) e
\]
\[
= (\nabla^h_{\partial^i} + h_{\partial^i} x^a e + h_{\partial^i} x^a e + h_{\partial^i} x^a e - h_{\partial^i} x^a e) e.
\]
Since for a normal variation we have

\[
\delta(g^{\partial^i} h_{\partial^i} x) = 2 h_{\partial^i} x^a e + g^{\partial^i} \delta h_{\partial^i} x,
\]
we obtain from (7.4)

\[
\delta(\frac{1}{n} g^{\partial^i} h_{\partial^i} x) = \frac{1}{n} (g^{\partial^i} \nabla^h_{\partial^i} x + K^h_{\partial^i \partial^j} B^j_{\partial^k} C^x_{\partial^i} e + h_{\partial^i} x^a e - h_{\partial^i} x^a e) e,
\]
where \( B^i = B^i_{\partial^i} g^{\partial^i} \).
In the sequel we suppose that \( m=n \) and the anti-invariant variation preserves \( f_\xi^x \). Since we have \( h^{cb}_c h^{cb}_b = 0 \) and \( \eta'_y = 0 \), (7.5) yields

**Proposition 7.1.** If \( m=n \) and an anti-invariant normal variation preserves \( f_\xi^x \), then we have

\[
\delta \left( \frac{1}{n} g^{cb}_c h^{cb}_b \xi^x \right) = \frac{1}{n} \left[ g^{cb}_c \nabla \xi^x_{cb} + K_{kji} C^k_j B^{ij} C^x_k \xi^x \right] \xi^x.
\]

**Corollary 7.1.** If \( m=n \) and an anti-invariant normal variation preserves \( f_\xi^x \), then it preserves the mean curvature vector if and only if

\[
g^{cb}_c \nabla \xi^x_{cb} + K_{kji} C^k_j B^{ij} C^x_k \xi^x = 0.
\]

Substituting (7.7) into

\[
\frac{1}{2} \Delta (\xi^x \xi_x) = \frac{1}{2} g^{cb}_c \nabla \xi^x_{cb} (\xi^x \xi_x) = (g^{cb}_c \nabla \xi^x_{cb}) \xi_x + (\nabla \xi^x \xi_x) (\nabla \xi^x),
\]

we find

\[
\frac{1}{2} \Delta (\xi^x \xi_x) = - K_{kji} C^k_j B^{ij} C^x_k \xi^x + (\nabla \xi^x \xi_x) (\nabla \xi^x).
\]

\( K_{kji} \) being covariant components of the curvature tensor of \( M^{2m} \).

If an anti-invariant submanifold \( M^n \) is compact and orientable, we find, from

(7.8)

\[
\int_M \left[ (\nabla \xi^x \xi_x) (\nabla \xi^x) - K_{kji} C^k_j B^{ij} C^x_k \xi^x \xi_x \right] dV = 0.
\]

Thus we have

**Theorem 7.1.** Suppose that \( m=n \) and an anti-invariant normal variation preserves \( f_\xi^x \) and the mean curvature vector. If \( M^n \) is compact and orientable and satisfies

\[
K_{kji} C^k_j B^{ij} C^x_k \xi^x \xi_x \leq 0,
\]

then the variation is parallel.

Suppose that the ambient Kaehlerian manifold \( M^{2m} \) is of constant holomorphic sectional curvature \( k \). Then we have

\[
K_{kji} = \frac{1}{4} k \left[ g^{kk}_k g^{ji}_j - g^{jj}_j g^{ki}_i + F^{kk}_k F^{ji}_j - F^{jj}_j F^{ki}_i - 2F^{kj}_k F^{ij}_i \right].
\]

Suppose also that a submanifold \( M'' \) of \( M^{2m} \) is anti-invariant. Then we have

\[
K_{kji} C^k_j B^{ij} C^x_k = \frac{1}{4} (m+3) k \xi^x.
\]

Thus we have, from Theorem 7.1,
PROPOSITION 7.2. Suppose that $M^{2m}$ is a Kaehlerian manifold of constant holomorphic sectional curvature $k \equiv 0$ and that $M^m$ is a compact orientable anti-invariant submanifold of $M^{2m}$. If an anti-invariant normal variation of $M^m$ preserves $f^k$ and the mean curvature vector, then the variation is parallel and $k \equiv 0$.

Suppose that the ambient Kaehlerian manifold has vanishing Bochner curvature tensor. Then we have (see [7])

\begin{equation}
(7.12) \quad K_{kijh} = - [g_{kh}L_{ji} - g_{jh}L_{ki} + L_{kh}g_{ji} - L_{jh}g_{ki} + F_{kh}M_{ji} - F_{jk}M_{hi} + M_{kh}F_{ji} - M_{jh}F_{ki} - 2(F_{kj}M_{ih} + M_{kj}F_{ih})],
\end{equation}

where

\begin{equation}
L_{ji} = - \frac{1}{2(m+2)} K_{ji} + \frac{1}{8(m+1)(m+2)} K_{gji},
M_{ij} = - L_{ji} F_{i}^j,
\end{equation}

$K_{ji}$ and $K$ being the Ricci tensor and the scalar curvature of $M^{2m}$ respectively.

Suppose also that a submanifold $M^m$ of $M^{2m}$ is anti-invariant. Then we have

\begin{equation}
(7.13) \quad K_{kijh} C^h_i B^j_k C^l_x = - [(m+3)L_{yx} + L_{gy} + 3L_{c} f^c_i f^h_x],
\end{equation}

where

\begin{equation}
L_{yx} = L_{ij} C^i_j C^j_x, \quad L = L_{ij} B^j_i, \quad L_{c} = L_{ij} B^j_{c}.
\end{equation}

But, on the other hand, we have

\begin{equation}
L_{c} f^i_j f^b_x = L_{ij} B^i_j B^j_i f^c_b f^c_x = L_{ij} F^i_j C^i_j F^c_i f^c_x = L_{yx},
\end{equation}

because of $L_{ij} F^i_j F^c_i = L_{c}$. Thus we have from (7.13)

\begin{equation}
(7.14) \quad K_{kijh} C^h_i B^j_k C^l_x = - [(m+6)L_{yx} + L_{gy}].
\end{equation}

Thus we have

PROPOSITION 7.3. Suppose that $M^{2m}$ is a Kaehlerian manifold with vanishing Bochner curvature tensor and that $M^m$ is a compact orientable anti-invariant submanifold of $M^{2m}$. If an anti-invariant normal variation of $M^m$ preserves $f^k$ and the mean curvature vector and

\begin{equation}
[(m+6)L_{yx} + L_{gy}] = 0,
\end{equation}

then the variation is parallel.

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REFERENCES


