THE EXTENDED SUM OF TWO RADICAL CLASSES

By R.E. Propes and A.M. Zaidi

In [6] Yu-Lee Lee and R.E. Propes defined the sum \( \alpha + \beta \) of two radical classes \( \alpha \) and \( \beta \) in a universal class \( W \) of associative rings as follows:

\[
\alpha + \beta = \{ R \in W : \alpha(R) + \beta(R) = R \}.
\]

They also found the necessary and sufficient conditions for this sum to be a radical class. The purpose of this paper is to extend this sum and to find conditions for which this extended sum will be a radical class.

We shall employ the following terms and notation throughout this paper:

- \( W \): a universal class of associative rings, i.e., a class of rings homomorphically closed and containing all ideals of all rings in \( W \).
- If \( \alpha \) is a class of rings then any ring in \( \alpha \) is called an \( \alpha \)-ring.
- If \( I \subseteq R \) (i.e., \( I \) is an ideal of a ring \( R \in W \)) then \( I \) is called an \( \alpha \)-ideal of \( R \).
- If \( R \in W \) has an \( \alpha \)-ideal which contains all \( \alpha \)-ideals of \( R \), it is called the \( \alpha \)-radical of \( R \), denoted by \( \alpha(R) \).
- A ring \( R \in W \) is \( \alpha \)-semisimple if \( R \) has no nonzero \( \alpha \)-ideals. The class of all \( \alpha \)-semisimple rings will be denoted by \( S(\alpha) \).

Recall [1] that a subclass \( \alpha \) of a universal class \( W \) of rings is called a radical class if and only if the following conditions are satisfied:

(a) \( \alpha \) is homomorphically closed,

(b) each ring \( R \in W \) has an \( \alpha \)-radical \( \alpha(R) \),

and (c) if \( R \in W \) then \( R/\alpha(R) \in S(\alpha) \).

DEFINITION. Let \( \alpha \) and \( \beta \) be two radical classes in \( W \). For each \( R \in W \) we set \( (\alpha \oplus \beta)(R) = \sum I \) where \( I \subseteq R \) and \( I/(\alpha(R) + \beta(R)) \in \alpha \cup \beta \).

DEFINITION. \( \alpha \oplus \beta = \{ R \in W : (\alpha \oplus \beta)(R) = R \} \).

NOTE. \( \alpha + \beta \subseteq \alpha \oplus \beta \) and \( S(\alpha + \beta) \subseteq \{ R \in W : (\alpha \oplus \beta)(R) = 0 \} \). (Recall [6] that \( S(\alpha + \beta) = S(\alpha) \cap S(\beta) \).

PROPOSITION 1. The class \( \alpha \oplus \beta \) is homomorphically closed.
PROOF. Let $R \in \mathfrak{A} \oplus \mathfrak{B}$ and $R/J$ be a homomorphic image of $R$. Now $R = \sum I$, where $I \subseteq R$ and $I/(\alpha(R) + \mathfrak{A}(R)) \subseteq \mathfrak{A} \cup \mathfrak{B}$. Thus $R/J = (\sum I)/J$. Let $\alpha(R/J) = K/J$ and $\mathfrak{A}/(R/J) = L/J$. Then $\alpha(R/J) + \mathfrak{A}(R/J) = K/J + L/J = (K + L)/J$. Let $(R/J)/\alpha(R/J) = (R/J)/(K + L) = (\sum I)/(K + L)$. Thus $\alpha(R/J) \subseteq K$ and $\mathfrak{A}(R/J) \subseteq L$ and hence $\alpha(R) + \mathfrak{A}(R) \subseteq K + L$. $(R/J)/(\alpha(R/J) + \mathfrak{A}(R/J)) = (R/J)/(K + L) = (\sum I)/(K + L)$. $(R/J)/\alpha(R/J) \subseteq (R/J)/(K + L)$ can be mapped homomorphically onto $(\sum I)/(K + L)$. By definition $I/(\alpha(R) + \mathfrak{B}(R)) \in \mathfrak{A} \cup \mathfrak{B}$. But $I/(\alpha(R) + \mathfrak{B}(R))$ can be mapped homomorphically onto $I/(I \cap (K + L)) \subseteq (I + K + L)/(K + L)$. Therefore $(I + K + L)/(K + L) \in \mathfrak{A} \cup \mathfrak{B}$. But $\sum I/(K + L) = \sum (I + K + L)/(K + L) = R/(K + L)$. By definition, $I/(\alpha(R) + \mathfrak{B}(R)) = \sum (P/J)$, where $P \subseteq R$ and $P/(K + L) \in \mathfrak{A} \cup \mathfrak{B}$. Now $(I + K + L)/J \leq R/J$ and $(I + K + L)/(K + L) \in \mathfrak{A} \cup \mathfrak{B}$. Thus $\sum (I + K + L)/J \subseteq (\alpha + \mathfrak{B})(R/J)$. But $\sum I/(K + L)/J = (\sum I)/(K + L)/J = R/J$.

Hence $(\alpha + \mathfrak{B})(R/J) = R/J$. i.e., $R/J \in \mathfrak{A} \cup \mathfrak{B}$.

PROPOSITION 2. $(\alpha \oplus \mathfrak{B})(R)$ is an $(\alpha \oplus \mathfrak{B})$-ideal of the ring $R$.

PROOF. Set $P = (\alpha \oplus \mathfrak{B})(R)$. Since $\alpha(R) + \mathfrak{B}(R) \subseteq P$, we have $\alpha(R) \subseteq P$ and $\mathfrak{B}(R) \subseteq P$. But $P \subseteq R$ so that $\alpha(R) = \mathfrak{B}(R)$ and $\alpha(R) = \mathfrak{B}(P)$. Let $I \subseteq R$ such that $I/(\alpha(R) + \mathfrak{B}(R)) \subseteq \mathfrak{A} \cup \mathfrak{B}$. Then by definition $I \subseteq P$. Moreover, $I/(\alpha(R) + \mathfrak{B}(R)) = I/(\alpha(P) + \mathfrak{B}(P)) \subseteq P/(\alpha(P) + \mathfrak{B}(P))$. Then, by definition, $I \subseteq (\alpha \oplus \mathfrak{B})(P)$. Thus $(\alpha + \mathfrak{B})(R) \subseteq (\alpha + \mathfrak{B})(P)$, and hence $(\alpha \oplus \mathfrak{B})(P) = (\alpha \oplus \mathfrak{B})(R)$.

PROPOSITION 3. Let $R \in W$. Then $(\alpha \oplus \mathfrak{B})(R)$ is the largest $(\alpha \oplus \mathfrak{B})$-ideal of $R$.

PROOF. Let $I \subseteq R$ and let $I \in \mathfrak{A} \oplus \mathfrak{B}$. Then $I = (\alpha \oplus \mathfrak{B})(I) = \sum J$, where $J \subseteq I$ and $J/(\alpha(I) + \mathfrak{B}(I)) \subseteq \mathfrak{A} \cup \mathfrak{B}$. Without loss of generality assume $J/(\alpha(I) + \mathfrak{B}(I)) \subseteq \alpha$. Then $J/(\alpha(I) + \mathfrak{B}(I)) \subseteq \alpha(I)/(\alpha(I) + \mathfrak{B}(I))$. But $J/(\alpha(I) + \mathfrak{B}(I)) \subseteq J/(\alpha(I) + \mathfrak{B}(I))$ and $\alpha$ is a radical class. Thus, by Theorem 1 [2], $\alpha(I)/(\alpha(I) + \mathfrak{B}(I)) \leq R/(\alpha(I) + \mathfrak{B}(I))$. Set $\alpha(I)/(\alpha(I) + \mathfrak{B}(I)) = K/(\alpha(I) + \mathfrak{B}(I))$. Then $K \subseteq R$ and $J \subseteq K$. Since $K/(\alpha(I) + \mathfrak{B}(I))$ is in $\alpha$ and can be mapped homomorphically onto $K/(K \cap (\alpha(R) + \mathfrak{B}(R))) \subseteq (K + \alpha(R) + \mathfrak{B}(R))/(\alpha(R) + \mathfrak{B}(R))$, we have $(K + \alpha(R) + \mathfrak{B}(R))/(\alpha(R) + \mathfrak{B}(R)) \subseteq \alpha \cup \mathfrak{A} \cup \mathfrak{B}$. Hence $K + \alpha(R) + \mathfrak{B}(R) \subseteq (\alpha \oplus \mathfrak{B})(R)$. Therefore $I = \sum J/(\alpha \oplus \mathfrak{B})(R)$.

THEOREM 1. Let $\alpha$ and $\mathfrak{B}$ be radical classes in a universal class $W$ of rings. If $S(\alpha) \subseteq \mathfrak{B}$ or $S(\mathfrak{B}) \subseteq \alpha$, then $\alpha \oplus \mathfrak{B}$ is a radical class.

PROOF. It suffices to show that $R/(\alpha \oplus \mathfrak{B})(R) \in S(\alpha \oplus \mathfrak{B})$. By definition,
(α+β)(R/α+β(R))=\sum I/(α+β)(R) where I\leq R and I/(J+K)\in α\cup β
where J/(α+β)(R)=α(R/(α+β)(R)) and K/(α+β)(R)=β(R/(α+β)(R)).
Without loss of generality, assume that S(α)\subseteq β. Now, (I/α(R))/(β(R)+I/α(R))\in β, because I/α(R)\subseteq S(α)\subseteq β. Since I/(α(R)+β(R))\cong (I/α(R))/(β(R))/α(R)) we have I/(α(R)+β(R))\in β\subseteq α\cup β.
Hence I\subseteq (α+β)(R), i.e., (α+β)(R/(α+β)(R))=0. Therefore R/(α+β)(R) is (α+β)-semisimple.

EXAMPLE 1. Let M be the class of fields of two elements. UM denotes the upper radical determined by M. By [3], every UM-semisimple ring is a subdirect sum of M-rings. Moreover, subdirect sums of fields of two elements are Boolean rings [7]. Hence R\in S(UM) which implies that R is a Boolean ring.

Let B be the class of all regular rings [R is a regular ring if for each a\in R, there exists an element x\in R such that axa=a]. As proved in [5], B is a radical class. Moreover, every Boolean ring is regular. Therefore, all Boolean rings are contained in B. Hence UM and B are two radical classes such that S(UM)\subseteq B and so satisfy the condition of the theorem.

THEOREM 2. Let α and β be two radical classes in W such that S(α)\subseteq β, S(β)\subseteq α and α\cap β=0. Then W=α+β.

PROOF. Let R\in W. Each of R/α(R) and R/β(R) can be mapped homomorphically onto R/(α(R)+β(R)). Now R/α(R)\in S(α)\subseteq B and R/β(R)\in S(β) \subseteq α. Therefore, R/(α(R)+β(R))\subseteq α\cap β=0. Hence R=α(R)+β(R) and so R\in W implies R\in α+β. Hence W=α+β. But we have already noticed that α+β\subseteq α+β. Hence α+β=α+β=W.

EXAMPLE 2. Let W be the universal class of rings whose additive groups are p-primary for some prime p. Let p be a prime and set T_p=\{R\in W: (R,+) is p-primary\}. Then T_p is a radical class [4]. Let Q=\bigcup_{q\neq p} T_q. We claim that Q is a radical class, in fact Q=U(T_p), upper radical class determined by T_p where

UT_p=\{R\in W: R/I\in T_p \forall I\in R\}.

Now R\in UT_p if and only if R\cong R/(0)\in T_q for some q\neq p if and only if R\in Q. Moreover, ST_p\subseteq Q, SQ\subseteq T_p and T_p\cap Q=0. Hence W=T_p+Q=T_p+Q.

University of Wisconsin-Milwaukee
REFERENCES