ON THE CATEGORY OF QUASI-UNIFORM SPACES

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1. Introduction

The category of topological spaces has recently attracted a great deal of attention. Herrlich and Strecker in [2] have considered coreflective subcategories in the category of topological spaces. Applying their techniques, we are able to characterize the coreflective subcategories in the category of quasi-uniform spaces. It is noted that the category of topological spaces is a retract of the category of quasi-uniform spaces and it is shown that the category of uniform spaces is coreflective in the category of quasi-uniform spaces.

DEFINITION 1.1. A quasi-uniform structure \( \mathcal{U} \) for a nonempty set \( X \) is a filter on \( X \times X \) satisfying:

1. \( \Delta = \{ (x, x) : x \in X \} \subseteq U \) for each \( U \) in \( \mathcal{U} \),
2. for each \( U \) in \( \mathcal{U} \) there exists a \( V \) in \( \mathcal{U} \) with \( V \circ V \subseteq U \).

DEFINITION 1.2. Let \( \mathcal{U} \) be a quasi-uniform structure on \( X \). Then let \( t_{\mathcal{U}} = \{ O \subseteq X : \text{if } x \in O \text{ then there exists } U \text{ in } \mathcal{U} \text{ with } x \in U[x] \subseteq O \} \).

It is easy to show that \( t_{\mathcal{U}} \) is a topology on \( X \). A quasi-uniform structure \( \mathcal{U} \) on \( X \) is said to be compatible with a topology \( t \) on \( X \) if \( t = t_{\mathcal{U}} \).

In [4], Pervin showed that the collection \( S = \{ O \times O \cup (X - O) \times X : O \in t \} \) formed a subbase for a quasi-uniform structure for a topological space \((X, t)\) which is compatible with \( t \). An excellent introduction to quasi-uniform spaces may be found in [3].

2. Category of quasi-uniform spaces

Let \( \mathcal{Q} \) denote the category of quasi-uniform spaces and quasi-uniformly continuous maps. \( \mathcal{Q}^\prime \) will denote the category of nonempty quasi-uniform spaces.

THEOREM 2.1. In the category \( \mathcal{Q} \),
(1) a morphism is a monomorphism if and only if it is one-to-one,
(2) a morphism is an epimorphism if and only if it is surjective,
(3) an isomorphism is a quasi-uniform space isomorphism,
(4) products are the quasi-uniform space products,
(5) coproducts are the disjoint quasi-uniform space union.

For a given set $X$ we let $\mathcal{U}_{X\times X}=\{X\times X\}$ and $\mathcal{U}_d=\{U\subseteq X\times X: d\subseteq U\}$.

**Theorem 2.2.** In the category $\mathcal{G}$,
(1) the only initial object is $(\phi, \mathcal{U}_{\phi\times\phi})$
(2) the terminal objects are of the form $([a], \mathcal{U}_d)$,
(3) has no zero object,
(4) the injective objects are precisely quasi-uniform spaces of the form $(X, \mathcal{U}_{X\times X})$, 
(5) the projective objects are precisely quasi-uniform spaces of the form $(X, \mathcal{U}_d)$.

Let $f: (X, \mathcal{U}) \rightarrow (Y, \mathcal{W})$ be surjective and set $\mathcal{V}$ equal to the supremum of all quasi-uniform structures on $Y$ for which $f$ is quasi-uniformly continuous. Then $f: (X, \mathcal{U}) \rightarrow (Y, \mathcal{V})$ is called a quotient map.

**Theorem 2.3.** In the category $\mathcal{G}$,
(1) $f: (X, \mathcal{U}) \rightarrow (Y, \mathcal{W})$ is an extremal monomorphism if and only if $(Y, \mathcal{W})$ is quasi-uniformly isomorphic to the subspace $f(X)$.
(2) $q : (X, \mathcal{U}) \rightarrow (Y, \mathcal{W})$ is an extremal epimorphism if and only if $(Y, \mathcal{W})$ is quasi-uniformly isomorphic to $(Y, \mathcal{V})$ where $\mathcal{V}$ is the quotient structure induced by $q$.

This theorem shows that the extremal monomorphisms in $\mathcal{G}$ are precisely the embedding maps while the extremal epimorphisms are the quotient maps.

Consider the morphism $f: (X, \mathcal{U}) \rightarrow (Y, \mathcal{V})$. Let $Z=f(X)$ and $\mathcal{W}$ be the restriction of $\mathcal{V}$ to $f(X)$. Let $f: X \rightarrow Z$ be defined by $f(x)=f(x)$ for each $x$ in $X$. Let $i: Z \rightarrow Y$ be the identity mapping. Then $f=if$ and $\mathcal{G}$ has the epi-mono factorization property. Moreover, if we let $\mathcal{W}$ be the quotient structure on $f(X)$ then $f$ is an extremal epimorphism and $\mathcal{G}$ thus has the extremal epi-mono factorization property. Since the category $\mathcal{G}$ is locally small we have by theorem 3 in [1] that $\mathcal{G}$ has the unique extremal epi-mono factorization property.

**Theorem 2.4.** The composite of two extremal epimorphisms in $\mathcal{G}$ is an extremal epimorphism. Thus $\mathcal{G}$ has the strong unique epi-mono factorization property.
PROOF. Let \((X, \mathcal{W}) \xrightarrow{f} (Y, \mathcal{W}') \xrightarrow{g} (Z, \mathcal{W})\) be given where \(f\) and \(g\) are extremal epimorphisms. Let \(\mathcal{P}\) be the supremum of all quasi-uniform structures on \(Z\) for which \(gf\) is quasi-uniformly continuous. Since \(gf\) is quasi-uniformly continuous with respect to \(\mathcal{W}\), we have that \(\mathcal{W} \leq \mathcal{P}\). Now consider:

\[
\begin{array}{c}
(X, \mathcal{W}) \xrightarrow{f} (Y, \mathcal{W}') \xrightarrow{g} (Z, \mathcal{W})
\end{array}
\]

\(i\) denotes the identity map and is quasi-uniformly continuous since \(\mathcal{W} \leq \mathcal{P}\). \(g: (Y, \mathcal{W}') \rightarrow (Z, \mathcal{P})\) is quasi-uniformly continuous since \(g^{-1}(\mathcal{W}')\) is a quasi-uniform structure on \(Y\) for which \(f\) is quasi-uniformly continuous. This follows from the fact that \(f\) is an extremal epimorphism and \(\mathcal{W}'\) is the strongest quasi-uniform structure on \(Y\) for which \(f\) is quasi-uniformly continuous. Since \(g: (Y, \mathcal{W}') \rightarrow (Z, \mathcal{W})\) is an extremal epimorphism and \(g=ig\) where \(i\) is a monomorphism, we must have that \(i: (Z, \mathcal{P}) \rightarrow (Z, \mathcal{W})\) is an isomorphism. Thus \(\mathcal{P} \leq \mathcal{W}\) and hence \(\mathcal{P} = \mathcal{W}\). Therefore \(gf\) is an extremal epimorphism.

Now since \(\mathcal{G}\) has the unique extremal epi-mono factorization property and the composite of extremal epimorphisms is an extremal epimorphism we have that \(\mathcal{G}\) has the strong unique extremal epi-mono factorization property.

**Theorem 2.5.** The constant morphisms in \(\mathcal{G}\) are precisely the constant maps.
The category \(\mathcal{G}'\), of nonempty quasi-uniform spaces, is constant generated.

**Proof.** The first statement is evident. Let \((X, \mathcal{W})\) and \((Y, \mathcal{W}')\) be objects in \(\mathcal{G}'\). Since \(Y\) is nonempty, there exists an element \(y\) in \(Y\). Define \(f: X \rightarrow Y\) by \(f(x) = y\) for each \(x\) in \(X\). Thus the set of morphisms from \(X\) to \(Y\) is nonempty.

Now let \(f, g: X \rightarrow Y\) be distinct morphisms. Then there is an element \(x\) in \(X\) with \(f(x) \neq g(x)\). Set \(Z = \{x\}\) and \(k: (Z, \mathcal{W}') \rightarrow (X, \mathcal{W})\) defined by \(k(x) = x\) is a constant morphism such that \(fk \neq gk\). Hence \(\mathcal{G}'\) is constant generated.

### 3. Coreflective subcategories

In this section each subcategory considered is assumed to be nontrivial. A subcategory \(\mathcal{U}\) of a category \(\mathcal{G}\) is said to be coreflective in \(\mathcal{G}\) if for each object \(X\) in \(\mathcal{G}\) there exists an object \(X_{\mathcal{U}}\) in \(\mathcal{U}\) and a morphism \(c_{\mathcal{U}}: X_{\mathcal{U}} \rightarrow X\), called the coreflective morphisms, such that for each object \(B\) in \(\mathcal{U}\) and morphism \(g: B \rightarrow X\) there exists a unique morphism \(h: B \rightarrow X_{\mathcal{U}}\) such that \(g = c_{\mathcal{U}}h\). \(\mathcal{U}\) is called epicoreflective if additionally each coreflective morphism is an epimorphism and it is...
called mono-coreflective if each coreflective morphism is a monomorphism.

For the convenience of the reader we state the following theorems found in [1].

**THEOREM A.** If $\mathcal{U}$ is a coreflective subcategory of a constant generated category $\mathcal{C}$ then $\mathcal{U}$ is both mono-coreflective and epi-coreflective.

**THEOREM B.** If $\mathcal{C}$ is a category which is
1. locally small,
2. has products,
3. has the extremal epi-mono factorization property,
and if $\mathcal{U}$ is a subcategory of $\mathcal{C}$ then the following statements are equivalent.
1. $\mathcal{U}$ is mono-coreflective in $\mathcal{C}$.
2. $\mathcal{U}$ is closed under the formation of coproducts and extremal quotient objects.

**THEOREM 3.1.** Let $\mathcal{U}$ be a subcategory of $\mathcal{C}$. The following statements are equivalent.
1. $\mathcal{U}$ is coreflective in $\mathcal{C}$.
2. $\mathcal{U}$ is mono-coreflective and epi-coreflective in $\mathcal{C}$.
3. $\mathcal{U}$ is closed under the formation of disjoint unions and quotient objects.

**PROOF.** (1)$\iff$(2) Let $\mathcal{U}$ be a coreflective subcategory of $\mathcal{C}$. Since we are considering only nontrivial subcategories we have that $\mathcal{U}$ is coreflective in $\mathcal{C} \iff \mathcal{U} \cap \mathcal{C}^\prime$ is coreflective in $\mathcal{C}^\prime$. Since $\mathcal{C}^\prime$ is constant generated by theorem 2.5, we have by theorem A that each coreflective subcategory of $\mathcal{C}^\prime$ must be both mono-coreflective and epi-coreflective. Hence each coreflective morphism $c_{\mathcal{U}}:X_{\mathcal{U}} \to X$ is one-to-one and onto.

(2)$\iff$(3) Since $\mathcal{C}$ satisfies the hypothesis for theorem B we have that a subcategory $\mathcal{U}$ of $\mathcal{C}$ is mono-coreflective if and only if (3) is satisfied, but if $\mathcal{U}$ is mono-coreflective then it is epi-coreflective by (1)$\implies$(2).

We now establish that for each subcategory $\mathcal{U}$ of $\mathcal{C}$ there exists a smallest coreflective subcategory $\mathcal{B}(\mathcal{U})$ containing $\mathcal{U}$ and moreover that the objects of $\mathcal{B}(\mathcal{U})$ are precisely the quotient objects of disjoint unions of members of $\mathcal{U}$.

The following theorems are found in [1].

**THEOREM C.** If $\mathcal{C}$ is a category which is
1. locally small,
2. has coproducts, and
3. has the extremal epi-mono factorization property
and if $\mathcal{U}$ is a subcategory of $\mathcal{C}$ then there exists a smallest mono-coreflective
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subcategory $A$ of $C$ containing $V$. Furthermore, if $C$ has the strong unique extremal epi-mono factorization property then the objects of $A$ are exactly all extremal quotient objects of coproducts of objects in $V$.

Let $\mathcal{C}(\mathcal{U})$ denote the smallest mono-coreflective subcategory in the category $C$ containing the subcategory $V$.

**Theorem D.** If $C$ is a category which

1. is locally small,
2. has coproducts,
3. has the strong unique extremal epi-mono factorization property,

and if $V$ is any subcategory of $C$ then each monomorphism in $C$ which is $V$-liftable is also $\mathcal{C}(\mathcal{U})$-liftable.

Using theorems C and D together with the fact that each coreflective subcategory in $C$ is mono-coreflective we have the following theorem.

**Theorem 3.2.** Let $\mathcal{U}$ be and subcategory of $\mathcal{C}$. Then

1. there exists a smallest coreflective subcategory $\mathcal{B}(\mathcal{U})$ containing $V$,
2. objects of $\mathcal{B}(\mathcal{U})$ are precisely the quotient objects of disjoint unions of objects in $V$,
3. each monomorphism in $C$ which is $\mathcal{U}$-liftable is $\mathcal{B}(\mathcal{U})$-liftable.

We now consider some interesting subcategories of $\mathcal{C}$ that are coreflective in $C$.

**Theorem 3.3.** The category of uniform spaces is a coreflective subcategory in $\mathcal{C}$, the category of quasi-uniform spaces.

**Proof.** Let $(X, \mathcal{U})$ be a quasi-uniform space. Now $(X, \mathcal{U} \vee \mathcal{U}^{-1})$ is a uniform space and the identity map $i: (X, \mathcal{U} \vee \mathcal{U}^{-1}) \to (X, \mathcal{U})$ is quasi-uniformly continuous. Let $(Y, \mathcal{V})$ be any uniform space and $f$ a morphism from $(Y, \mathcal{V})$ to $(X, \mathcal{U})$. Define $\tilde{f}: (Y, \mathcal{V}) \to (X, \mathcal{U} \vee \mathcal{U}^{-1})$ by $\tilde{f}(y) = f(y)$ for each $y$ in $Y$. Now $f = i \tilde{f}$ and $\tilde{f}$ is unique. We must show that $\tilde{f}: (Y, \mathcal{V}) \to (X, \mathcal{U} \vee \mathcal{U}^{-1})$ is quasi-uniformly continuous. It suffices to show that $\tilde{f}^{-1}(U \cup \mathcal{U}^{-1}) \subseteq \mathcal{V}$ for each $U \subseteq \mathcal{U}$. Let $U \subseteq \mathcal{U}$, then $\tilde{f}^{-1}(U) \subseteq \mathcal{V}$ and hence $\tilde{f}^{-1}(U) \subseteq \mathcal{V}$. Since $\mathcal{V}$ is a uniform structure, there exists a symmetric $V \subseteq \mathcal{V}$ with $V \subseteq \tilde{f}^{-1}(U)$. Thus $V \subseteq \tilde{f}^{-1}(U^{-1})$ and $\tilde{f}^{-1}(U \cup \mathcal{U}^{-1}) \subseteq \mathcal{V}$. Hence $f$ is quasi-uniformly continuous.

**Theorem 3.4.** The category of fine quasi-uniform spaces is coreflective in the category of quasi-uniform spaces.
The proof of this theorem is natural. A quasi-uniform space \((X, \mathcal{U})\) is called saturated if \(\mathcal{U}\) is closed under arbitrary intersections. A space is saturated if and only if the structure \(\mathcal{U}\) has a base consisting of a single set.

**Theorem 3.5.** The category of saturated quasi-uniform spaces is coreflective in \(\mathcal{C}\).

**Proof.** Let \((X, \mathcal{U})\) be a quasi-uniform space. Set \(S = \cap \{U : U \in \mathcal{U}\}\). Then \(S = S = S\) and \([S]\) forms a base for a saturated quasi-uniform structure \(\mathcal{S}\). Let \(i : (X, \mathcal{S}) \to (X, \mathcal{U})\) denote the identity map, then \(i\) is quasi-uniformly continuous. Suppose that \((Y, \mathcal{V})\) is a saturated space and \(f : (Y, \mathcal{V}) \to (X, \mathcal{U})\) a morphism in \(\mathcal{C}\). Now define \(\bar{f} : (Y, \mathcal{V}) \to (X, \mathcal{S})\) by \(\bar{f}(y) = f(y)\) for each \(y\) in \(Y\). Then \(f = i \bar{f}\) and \(\bar{f}\) is unique. To see that \(\bar{f}\) is quasi-uniformly continuous, note that \(\mathcal{V}\) is generated by a base \(\{T\}\). Then for each \(U\) in \(\mathcal{U}\) we have \(T \subseteq f^{-1}(U)\) and thus \(T \subseteq \bar{f}^{-1}(U)\). Therefore \(T \subseteq \bar{f}^{-1}(U) : U \in \mathcal{U}\) = \(\bar{f}^{-1}(U) \cap \{U : U \in \mathcal{U}\}\) = \(f^{-1}(S)\). Hence \(\bar{f}\) is quasi-uniformly continuous.

### 4. Special functors

In this section we consider two natural functors.

**Theorem 4.1.** The category of topological spaces is a retract of the category \(\mathcal{O}\), the category of quasi-uniform spaces.

**Proof.** Let \(\mathcal{S}\) denote the subcategory of \(\mathcal{O}\) of quasi-uniform spaces with the Pervin quasi-uniform structure. \(\mathcal{T}\) will denote the category of topological spaces and continuous maps. Let \(T : \mathcal{O} \to \mathcal{T}\) be the natural functor from a quasi-uniform space to the underlying topological space. Let \(P : \mathcal{T} \to \mathcal{S}\) be the functor that associates with each topological space the corresponding Pervin quasi-uniform space. Now \(\mathcal{S}\) is a full subcategory of \(\mathcal{O}\), and the functor \(PT : \mathcal{O} \to \mathcal{S}\) is the identity functor on \(\mathcal{S}\). Also \(TP : \mathcal{T} \to \mathcal{T}\) is the identity functor on \(\mathcal{T}\). Hence \(\mathcal{S}\), a full subcategory of \(\mathcal{O}\), is a retract of \(\mathcal{O}\) and \(\mathcal{S}\) and \(\mathcal{T}\) are isomorphic.

Define \(R : \mathcal{O} \to \mathcal{O}\) by \((X, \mathcal{U}) \to (X, \mathcal{U}^{-1})\). If \(f : (X, \mathcal{U}) \to (Y, \mathcal{V})\) is a morphism in \(\mathcal{O}\) then define \(R(f)(x) = f(x)\) for each \(x\) in \(X\). \(R\) will be called the conjugate functor on \(\mathcal{O}\).

**Theorem 4.2.** \(R\) is a functor on \(\mathcal{O}\) such that \(R \circ R\) is the identity functor on \(\mathcal{O}\). The fixed points of \(R\) are precisely the uniform spaces.

**Proof.** Let \(f : (X, \mathcal{U}) \to (Y, \mathcal{V})\) be a morphism in \(\mathcal{O}\), and let \(V^{-1} \in \mathcal{V}^{-1}\). Then
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$V \in \mathcal{V}$ and there exists a $U \in \mathcal{U}$ with $U \subseteq f^{-1}(V)$. Thus $U^{-1} \subseteq f^{-1}(V^{-1})$ and $f: (X, \mathcal{U}^{-1}) \rightarrow (Y, \mathcal{V}^{-1})$ is quasi-uniformly continuous. The other properties are easy to verify and $R$ is indeed a functor on $\mathcal{C}$. That $R \circ R$ is the identity on $\mathcal{C}$ is evident. Now $R((X, \mathcal{U})) = (X, \mathcal{U'})$ if and only if $\mathcal{U'} = \mathcal{U}^{-1}$. Thus the fixed points of $R$ are precisely the uniform spaces.

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