ON THE NUMBER OF ISOMORPHISM CLASSES OF CERTAIN TYPES OF ACTIONS AND BINARY SYSTEMS

By Joseph Neggers

Introduction

While in [3] we counted a.o. the number $K(m,n)$ of isomorphism classes of actions $Y_X$ defined on sets $X \times Y$ with $|X|=m$ and $|Y|=n$, and the number of isomorphism classes of binary systems $B(n)$ defined on sets $X$ with $|X|=n$, in [2] we counted a variety of classes of finite posets. Some of the counting techniques developed there have ready applications and variations which are useful in a variety of other contexts. By way of illustration we shall develop some formulas for special classes of actions and binary systems. We begin with some definitions and a listing of information which we shall assume as known.

An action of $X$ on $Y$ is a function $f: X \times Y \rightarrow Y$. We shall usually denote actions by $Y_X$ and $f(x,y)=xy$. The set $X$ is considered to be the set of scalars.

Another way to view actions is as follows. Let $Y$ be the set of vertices of a polychromatic directed graph and let $X$ be a set of colours. If $xy_1=y_2$, then we envisage this as representing an arrow of colour $x$ as proceeding from $y_1$ to $y_2$. In this sense actions are special types of graphs i.e., a polychromatic directed graph is an action provided there is precisely one arrow of each colour $x \in X$ departing from each vertex $y \in Y$. More generally one gets involved with partial actions, where $f$ has domain a subset of $X \times Y$ and various other refinements.

As with modules, if $A_X$ and $B_X$ are actions on the same set, then $\phi: A \rightarrow B$ is a homomorphism provided $\phi(xa)=\phi(x)\phi(a)$). A homomorphism $\phi$ which is also a bijection is an isomorphism.

For actions we define the following operations, which correspond to rather natural constructions in a variety of cases.

Sum: $Y_X+V_U=T_S$, where $T=Y \cup V$ (disjoint union), $X=(X \cup U)$ (disjoint union) and $xv=v$, $uv=y$. If $xy=y$ and $uv=v$ for all $y$ and for all $v$ respectively, then we shall identify $x$ and $u$.

Ordinal sum: $Y_X \oplus V_U=T_S$, where $T=Y \cup V$ (disjoint union), $S=X \cup V \cup U$.
(disjoint union), \(xv = v, vy = v, uy = y, vu' = v'\). If \(xy = y\) and \(uv = v\) for all \(y\) and for all \(v\) respectively, then we shall identify \(x\) and \(u\).

Product: \(Y_xV_y = T_{xy}\), where \(T = Y \times V\), \(S = X \times U\), \((x, u)(y, v) = (xy, uv)\).

Selective product: \(Y_xV_y = T_{xy}\), where \(T = Y \times V\), \(S = X \cup U\) (disjoint union), \((x, y)(v, v') = (xy, v')\). If \(xy = y\) and \(uv = v\), \(x\) and \(u\) are identified.

An action is sum-primitive (S-primitive), OS-primitive, P-primitive, SP-primitive respectively, if it cannot be written as a sum, ordinal sum, product or selective product respectively.

An action \(Y_x\) is faithful if \(xy = x' y\) for all \(y \in Y\) implies \(x = x'\). Hence, if \(Y_x\) and \(V_y\) are faithful, then it is easily seen that their sum, ordinal sum, product and selective product are also faithful. An action \(Y_x\) is unitary provided for some \(x \in X\), \(xy = y\) for all \(y \in Y\).

For binary systems \(B\), we define the following operations:
- \(LS(B_1, B_2) = B, B = B_1 \cup B_2\) (disjoint union) and \(b_i \in B_i\) implies \(b_1 b_2 = b_1, b_2 b_1 = b_1\);
- \(RS(B_1, B_2) = B, B = B_1 \cup B_2\) (disjoint union) and \(b_i \in B_i\) implies \(b_1 b_2 = b_1, b_2 b_1 = b_2\);
- \(MS(B_1, B_2) = B, B = B_1 \cup B_2\) (disjoint union) and \(b_i \in B_i\) implies \(b_1 b_2 = b_2, b_2 b_1 = b_2\);
- \(RMS(B_1, B_2) = B, B = B_1 \cup B_2\) (disjoint union) and \(b_i \in B_i\) implies \(b_1 b_2 = b_1, b_2 b_1 = b_1\).

We shall call these operations left sum, right sum, middle sum and reverse middle sum respectively. Again, these various sums correspond to a variety of rather natural constructions. There are of course a host of other possibilities, but we shall only look at these. Again we have a notion of primitivity, with \(B\) \(LS\)-primitive if it is not of the form \(B = LS(B_1, B_2)\), with the notions \(RS\)-primitive, \(MS\)-primitive and \(RMS\)-primitive defined similarly. Finally, a binary system is primitive if it is \(LS\)-, \(RS\)-, \(MS\)- and \(RMS\)-primitive simultaneously.

If \(B\) is a binary system such that \(B = B_1 \cup \ldots \cup B_k\), where \(B_i \neq \emptyset\), \(B_i \cup B_j \leq B_i, k \geq 2\), and \(B_i \cup B_j\) is one of \(LS(B_i, B_j)\), \(RS(B_i, B_j)\), \(MS(B_i, B_j)\) or \(RMS(B_i, B_j)\), then \(B\) is decomposable. \(B\) is indecomposable if it is not decomposable.

If we take for granted the numbers \(K(m, n)\) and \(B(n)\), then in this paper we shall essentially determine the number of isomorphism classes of the various types of primitive actions and the various types of primitive binary systems. We shall also determine the number of isomorphism classes of indecomposable binary systems. We recall that \(T(k, n)\), the number of actions \(Y_x\) such that \(X \subseteq Y\), \(|X| = k, |Y| = n\) has also been determined in [3].
On the Number of Isomorphism Classes of Certain Types of Actions and Binary Systems

On the number of faithful S-primitive actions.

If \( T(n) = \sum_{k=1}^{n} T(k, n) \), then \( T(n) \) is determined and represents the total number of actions \( Y_X \) with \( X \subseteq Y \), \( |Y| = n \) and \( |X| \geq 1 \). Since \( X \subseteq Y \), the action \( Y_X \) is faithful, and conversely if \( Y_X \) is a faithful action, then \( x \rightarrow f_x \), where \( f_x y = xy \), is an injection of \( X \) into \( Y \), so that in fact the condition that \( Y_X \) is faithful is essentially equivalent to our taking \( X \) to be a subset of \( Y \). Therefore \( T(n) \) represents the total number of faithful actions \( Y_X \) with \( |Y| = n \).

Let \( \{ \alpha_i \}_{i \in \omega} \) denote the collection of finite \( S \)-primitive faithful actions. Let \( P = \{ \sum n_i \alpha_i \}_{n_i \in \{0, 1, 2, \ldots\}} \) denote the collection of finite linear combinations of finite \( S \)-primitive faithful actions. Let \( f^*(\alpha_i) = |Y| \) if \( \alpha_i = Y_X \). Let \( f(\sum n_i \alpha_i) = \sum n_i f^*(\alpha_i) \). Let \( T^*(n) = |\{ \alpha_i | f^*(\alpha_i) = n \}| \), \( T'(n) = |\{ \sum n_i \alpha_i | f(\sum n_i \alpha_i) = n \}| \). Obviously \( T'(n) \) is identical to the number \( T(n) \) defined above. We're interested in determining the number \( T^*(n) \) of \( S \)-primitive faithful actions \( Y_X \) such that \( |Y| = n \).

If \( T_0(k, n) \) denotes the number of faithful unitary actions \( Y_X \) with \( |X| = k \), \( |Y| = n \), then by adjoining identity maps we find that:

1. \( T(k, n) - T_0(k, n) = T_0(k+1, n) \).

Hence, if \( T_0(n) = \sum T_0(k, n) \), then:

2. \( T(n) = \sum_k T_0(k, n) + \sum_k T_0(k+1, n) = 2T_0(n) - T_0(1, n) = 2T_0(n) - 1 \).

Also, we have:

3. \( T_0(k, n) = \sum_{i=0}^{k} (-1)^i T(k+i, n) \), so that the numbers \( T_0(n) \) and \( T_0(k, n) \) are determined.

If \( T_0^*(n) \) and \( T_0^*(k, n) \) denote the number of \( S \)-primitive faithful unitary actions \( Y_X \) with \( |Y| = n \) (resp. \( |X| = k \), \( |Y| = n \)), then since adding or removing an identity map will not affect the \( S \)-primitivity of an action, we have:

4. \( T_0^*(k, n) + T_0^*(k+1, n) = T^*(k, n) \), the number of \( S \)-primitive faithful actions \( Y_X \) with \( |X| = k \), \( |Y| = n \).

Hence, it follows that:

5. \( T^*(n) = \sum_k T^*(k, n) = 2T_0^*(n) - T_0^*(1, n) = 2T_0^*(n) - 1 \), as in formula (2) for \( T(n) \) and \( T_0(n) \).

Summing is a commutative operation, i.e., \( Y_X + V_U = V_U + Y_X \) in the sense of
identical actions and hence certainly if we use \( \cong \) to denote isomorphic actions.

In the following we use constructions also discussed in [2], so for other examples we refer the reader to that paper.

Let \( W = \{1, 2, \ldots \} \) be the set of positive integers. Define products \( W^k \times W^k \to W \) as follows:

\[
(\epsilon_1, \ldots, \epsilon_k) \cdot (n_1, \ldots, n_k) = e_1 n_1 + \cdots + e_k n_k \quad \text{and} \quad (\epsilon_1, \ldots, \epsilon_k) \times (n_1, \ldots, n_k) = n_1^{(\epsilon_1)} \cdots n_k^{(\epsilon_k)},
\]

where \( n^{(\epsilon)} = \binom{n}{\epsilon} \binom{\epsilon}{\epsilon} + \cdots + \binom{n}{\epsilon-1} \binom{\epsilon-1}{\epsilon-1} = \binom{n+\epsilon-1}{\epsilon} \).

Thus \( n^{(\epsilon)} \) is also equal to the number of ordered partitions of \( n-1 \) into \( \epsilon+1 \) non-negative integers.

If \( f : W \to W \) is any function whatsoever, we define

\[
(\epsilon_1, \ldots, \epsilon_k) \times f(n_1, \ldots, n_k) = (\epsilon_1, \ldots, \epsilon_k) \times (f(n_1), \ldots, f(n_k)).
\]

Thus, if \( n_1 > n_2 > \cdots > n_k \), then \( (\epsilon_1, \ldots, \epsilon_k) \times T_\epsilon(n_1, \ldots, n_k) \) is the number of faithful unitary actions having \( e_1 + \cdots + e_k \) \( S \)-primitive faithful unitary components, of which precisely \( e_i \) components have \( n_i \) elements.

That this is indeed the case can be seen as follows. Suppose that we consider all faithful unitary actions having \( e \) \( S \)-primitive faithful unitary components each containing \( n \) elements. Then, if we assume that there are \( k \)-non-isomorphic types present, then these can be distributed in \( p_k(\epsilon) = \binom{\epsilon-1}{k-1} \) ways, where \( p_k(\epsilon) \) is the number of ordered partitions of \( \epsilon \) into \( k \) positive integers, each such partition corresponding to a distinct distribution of these \( k \) types. Now, if \( T_\epsilon(n) = m \), then these \( k \) types can be selected in \( \binom{m}{k} \) ways, disregarding order. Hence the total number of actions which can be constructed from \( k \) types is \( \binom{m}{k} \binom{\epsilon-1}{k-1} \). Summing over \( k \), we find that we end up with \( m^{(\epsilon)} \) possibilities. The more general case follows at once, since actions having different cardinalities cannot be isomorphic, i.e., the product rule holds in that situation when considering the decomposition of the faithful unitary action into its \( e_i \) components each containing \( n_i \) elements for \( i = 1, \ldots, k \).

We shall call a vector \( \vec{n} \) in \( W^k \) c\( a \)\( s \)\( c \)\( i \)\( n \)g\( i \)\( d \) if \( \vec{n} = (n_1, \ldots, n_k) \) with \( n_1 > n_2 > \cdots > n_k \).

A bipartition of \( m \) is a pair \((\vec{\epsilon}, \vec{n}) \in W^k \times W^k \) for some \( k \) such that \( \vec{n} \) is c\( a \)\( s \)\( c \)\( i \)\( n \)g\( i \)\( d \) and such that \( \vec{\epsilon} \cdot \vec{n} = m \).

Hence, if we consider the sum over all bipartitions of \( m \sum_{\vec{\epsilon} \cdot \vec{n} = m} \vec{\epsilon} x T_\vec{\epsilon} \vec{n} \), then this number is \( T_0(m) \).
Now let $g_1(X_1) = X_1$, $\bar{e} \otimes \bar{N} = g_{N_1}(x_{N_1}, \ldots, x_{N_1})^{(e_1)} \ldots g_{N_e}(x_{N_e}, \ldots, x_{N_e})^{(e_e)}$ and $g_n(X_1, \ldots, X_n) = X_n - \sum_{\bar{e}, \bar{N}=n}^{*} \bar{e} \times \bar{N}$ where $\sum^{*}$ indicates that we delete the bipartition $(1) \cdot (n) = n$, and where $\rho(x)^{(e_i)} = \left(\frac{\rho(x) + e - 1}{e}\right) = 1/e! (\rho(x) + e - 1) (\rho(x) + e - 2) \ldots (\rho(x) + 1) \rho(x)$ for $\rho(x) = \rho(x_1, \ldots, x_p)$ any polynomial in any finite number of indeterminates. Then it follows that

$$(8) \quad T_0^*(n) = g_n(T_0(1), \ldots, T_0(n)),$$

as can be seen by simple substitution of the formulas for $T_0(i)$, $1 \leq i \leq n$ into $g_n(x_1, \ldots, x_n)$. Indeed, if $f$ and $g$ are functions which are related by an equation:

$$(9) \quad f(m) = \sum_{\bar{e}, \bar{n} = m} \bar{e} \bar{x} \bar{y},$$

then it is also true that:

$$(10) \quad h(n) = g_n(f(1), \ldots, f(n)).$$

The polynomials $g_n(x_1, \ldots, x_n)$ can be computed recursively and in [2] the first several ones are given. As we shall see below, this type of process has many variants and applications, in this paper we give only a few examples.

In order to determine $T_0^*(k, n)$ we proceed in a similar fashion. If $w$ is the ordered set $\{1, 2, 3, \ldots\}$ then we turn $w^2$ into an ordered set by letting $a = (a_1, a_2) < b = (b_1, b_2)$ if $a_1 + a_2 < b_1 + b_2$ or if $a_1 + a_2 = b_1 + b_2$ and $a_1 < b_1$. Thus, the number of elements preceding any given element is finite. If $\bar{a} = (a_1, \ldots, a_t), a_i \in w^2$, say $\bar{a}$ is cascading if $a_1 > a_2 > \ldots > a_t = (1, 1)$. For $\bar{e} \in w^t, \bar{a} \in (w^2)^t$, let $\bar{e} \cdot \bar{a} = (\sum e_i a_i) + 1 - t^*, \sum e_i a_i, a_i = (a_{i1}, a_{i2}), t^* = e_1 + \ldots + e_t$.

Then, by an argument very similar to that used above we establish that:

$$(11) \quad T_0(k, n) = \sum_{\bar{e}, \bar{a} = (k, n)}^{t^*} \bar{e} \bar{a},$$

where

$$\bar{e} \times T_0^* \bar{a} = (e_1, \ldots, e_t) \times (T_0^*(a_{11}, a_{12}), \ldots, T_0^*(a_{1t}, a_{1t})).$$

As before in the sum in (11), $\bar{a}$ runs over all cascading vectors. In particular, $(1) \cdot (k, n) = (k, n)$ and $(1) \times T_0^*(k, n) = T_0^*(k, n) = T_0^*(k, n)$.

Again, if we define the polynomials $g_{(k, n)}(x_{11}, \ldots, x_{kn})$ inductively by the scheme

$$(13) \quad g_{(1, 1)}(x_{11}) = x_{11}, \quad g_{(k, n)}(x_{11}, \ldots, x_{kn}) = x_{kn} - \sum_{\bar{e}, \bar{a} = (k, n)}^{*} \bar{e} \otimes \bar{a},$$

where $\sum^*$ indicates that we delete the bipartition $(1) \cdot (k, n) = (k, n)$ and where
(14) \[ \bar{\epsilon} \otimes \bar{a} = g_{(a_1, a_2)} \left( x_{11}, \ldots, x_{a_1, a_2}, (e_i) \right) \cdots g_{(a_1, a_2)} \left( x_{11}, \ldots, x_{a_t, a_2}, (e_i) \right). \]

It follows that we can invert formula (11) to obtain

(15) \[ T_0^* (k, n) = g_{(k, n)} (T_0 (1), \ldots, T_0 (k, n)). \]

From (14) and (15) we have also determined \( T^* (k, n) \), while (5) and (9) give us the number \( T^* (n) \).

**On the number of faithful OS-primitive actions**

The next item on the agenda is to deal with ordinal sums. Here, since for OS-primitive actions \( Y_X \) and \( V_U, Y_X \otimes V_U = V_U \otimes Y_X \) if and only if \( Y_X = V_U \), we have the extreme non-commutative situation as in ordinal sums of posets. The solutions follow the pattern established above using different functions.

Let \( F_0^* (k, n) \) denote the number of OS-primitive faithful unitary actions \( Y_X \) with \( |X| = k, |Y| = n \). Let \( F_0^* (n) = \sum F_0^* (k, n) \).

From the non-commutativity of ordinal sums in the sense described above, we establish in a straightforward manner that

(16) \[ T_0^* (n) = \sum_{\bar{e} \in F_0^* \bar{N}} \bar{e} \otimes \bar{N}, \] where

(17) \[ (e_1, \ldots, e_t) \otimes (N_1, \ldots, N_t) = N_1^{e_1} \cdots N_t^{e_t} \frac{(e_1 + \cdots + e_t)}{e_1, \ldots, e_t}, \]

and

(18) \[(e_1, \ldots, e_t) \otimes (f(N_1), \ldots, f(N_t)). \]

Here as usual in (16) we sum only over bipartitions.

To invert formula (16) we define polynomials \( G_n (x_1, \ldots, x_n) \) inductively by

(19) \[ G_n (x_1, \ldots, x_n) = x_n - \sum_{\bar{e} \in F_0^* \bar{N}} \bar{e} \otimes \bar{N}, \]

(20) \[ (e_1, \ldots, e_t) \otimes (N_1, \ldots, N_t) = G_{N_1} (x_1, \ldots, x_{N_1}) \cdots G_{N_t} (x_1, \ldots, x_{N_t}) \frac{(e_1 + \cdots + e_t)}{e_1, \ldots, e_t}. \]

Again in (19) \( \sum^* \) indicates that the bipartition \( (1) \cdot (n) = n \) is deleted. It follows that the polynomials \( G_n (x_1, \ldots, x_n) \) can be computed recursively and that

(21) \[ F_0^* (n) = G_n (T_0 (1), \ldots, T_0 (n)). \]

In order to compute \( F_0^* (k, n) \) we take a product on \( w^2 \) defined as follows:

(22) \[ \bar{e} \cdot \bar{a} = (a_1, \ldots, e_i) \cdot \left( (a_1, a_{12}), \ldots, (a_t, a_{12}) \right) \]

\[ = (\sum e_i (a_1 + a_{12}) + 1 - a_{12} - t^*, \sum a_i a_{12}). \]

A typical argument shows that:

(23) \[ T_0 (k, n) = \sum_{\bar{e} \in F_0^* \bar{N}} \bar{e} \otimes \bar{N}, \] where
On the Number of Isomorphism Classes of Certain Types of Actions and Binary Systems

(24) \( \tilde{e} \wedge_F \tilde{a} = F_0^*(a_{11}, a_{12}) \cdots \cdot F_0^*(a_{r1}, a_{r2}) \tilde{e}^{e_1 + \cdots + e_r} \).

Thus if we take \( G_{(1,1)}(x_{11}) = x_{11} \) and

\[
\sum_{\tilde{e} \in a_{1n}} \tilde{e} \odot \tilde{a},
\]

(25) \( G_{(k,n)}(x_{11}, \ldots, x_{kn}) = x_{kn} - \sum_{\tilde{e} \in a_{1n}} \tilde{e} \odot \tilde{a}, \)

(26) \( \tilde{e} \odot \tilde{a} = G(a_{1i}, a_{in})(x_{11}, \ldots, x_{1i}) \tilde{e}^{e_1 + \cdots + e_i} \).

With \( \sum_{\tilde{e} \in a} \) running over all bipartitions except \( (1) \cdot (k, n) = (k, n) \) we have the inversion we need. Hence

(27) \( F_0^*(k, n) = G_{(k,n)}(T_0(1,1), \ldots, T_0(k,n)). \)

If we let \( F^*(n) \) denote the number of faithful OS-primitive actions \( Y_X \) such that \( |Y| = n \), then as before we observe that adding an identity map does not change OS-primitivity, and hence as above \( F^*(n) = 2F_0^*(n) - 1 \). Also, if \( F^*(k, n) \) denotes the number of faithful OS-primitive actions \( Y_X \) such that \( |Y| = n \), then as in (4) we have a relation \( F_0^*(k, n) + F_0^*(k+1, n) = F^*(k, n) \). Thus the quantities \( F^*(n) \) and \( F^*(k, n) \) have also been determined.

**On the number of faithful P-primitive actions**

For products we are back in the commutative situation. Also, since \( Y_X V_U \) is unitary and faithful if and only if \( Y_X \) and \( V_U \) are both unitary and faithful, then we may compute certain coefficients without first passing to unitary actions. Thus let \( P_0^*(k, n) \) denote the number of faithful unitary \( P \)-primitive actions \( Y_X \) such that \( |Y| = n \), \( |X| = k \). Let \( P_0^*(n) \) denote the number of faithful unitary \( P \)-primitive actions \( Y_X \) such that \( |Y| = n \), \( |X| = k \). Let \( P^*(k, n) \) denote the number of faithful \( P \)-primitive actions \( Y_X \) such that \( |Y| = n \) and \( |X| = k \). Finally let \( P^*(n) \) denote the number of faithful \( P \)-primitive actions \( Y_X \) such that \( |Y| = n \). Obviously, \( P_0^*(n) = \sum_k P_0^*(k, n) \).

Now, we use a scalar product

(28) \( (e_1, \ldots, e_r) \cdot (N_1, \ldots, N_r) = N_1^{e_1} \cdots N_r^{e_r} \), and the vector product

(29) \( (e_1, \ldots, e_r) \times (N_1, \ldots, N_r) = N_1^{(e_1)} \cdots N_r^{(e_r)} \).

A bipartition \( \tilde{e} \cdot \tilde{N} = n \) requires \( \tilde{N} \) to be a cascading vector. In a straightforward manner we find

(30) \( T_0(n) = \sum_{\tilde{e} \cdot \tilde{N} = n} \tilde{e} \times \tilde{N}, \quad T(n) = \sum_{\tilde{e} \cdot \tilde{N} = n} \tilde{e} \times \tilde{N}. \)
Here the summation runs over bipartitions \( e \cdot \overline{N} = n \) as usual.

If we let \( h_1(x_1) = x_1 \) and

\[
(31) \quad h_n(x_1, \ldots, x_n) = x_n - \sum_{e \cdot \overline{N} = n}^{\times} e \otimes \overline{N},
\]

with \( \sum^{\times} \) indicating that \( (1) \cdot (n) = n \) is excluded, then it follows that

\[
(32) \quad P_0^*(n) = h_n(T_0(1), \ldots, T_0(n)), \quad P^*(n) = h_n(T(1), \ldots, T(n)).
\]

For \( P_0^*(k, n) \) and \( P^*(k, n) \) we work with bipartitions of elements in \( w^2 \) and we use the scalar product:

\[
(33) \quad e \cdot \overline{a} = (e_1, \ldots, e_i) \cdot ((a_{11}, a_{12}), \ldots, (a_{t1}, a_{t2}))
\]

\[
= ((e_1, \ldots, e_i) \cdot ((a_{11}, a_{12}), \ldots, (a_{t1}, a_{t2}))
\]

\[
= (\prod_{i=1}^{t} a_i e_i) = (a_{11} e_1, \ldots, a_{t2} e_t).
\]

Another argument of the standard type and we find

\[
(34) \quad T_0(k, n) = \sum_{e \cdot \overline{a} = (k, n)}^{\times} e \otimes \overline{a}, \quad T(k, n) = \sum_{e \cdot \overline{a} = (k, n)}^{\times} e \otimes \rho^* a.
\]

Thus to invert we define \( h_{(1,1)}(x_{11}) = x_{11} \) and

\[
(35) \quad h_{(k, n)}(x_{11}, \ldots, x_{kn}) = x_{kn} - \sum_{e \cdot \overline{a} = (k, n)}^{\times} e \otimes \overline{a}.
\]

with \( (1) \cdot (k, n) = (k, n) \) deleted in \( \sum^{\times} \) as usual. It follows that

\[
(36) \quad P_0^*(k, n) = h_{(k, n)}(T_0(1, 1), \ldots, T_0(k, n))
\]

\[
P^*(k, n) = h_{(k, n)}(T(1, 1), \ldots, T(k, n)).
\]

**On the number of faithful \( SP \)-primitive actions**

For selective products we are again dealing with a commutative situation.

Because of the definition we work with unitary actions and numbers \( S_0^*(k, n) \) and \( S_0^*(n) \), where the first number is the number of faithful unitary \( SP \)-primitive actions \( Y_X \) with \( |Y| = n \) and \( |X| = k \), and where the second number is the number of faithful unitary \( SP \)-primitive actions \( Y_X \) with \( |Y| = n \).

If we define \( \hat{e} \cdot \overline{N} \) as in (28) and \( \hat{e} \times \overline{N} \) as in (29), then

\[
(37) \quad T_0(n) = \sum_{e \cdot \overline{N} = n}^{\times} e \times S_0^* \overline{N}
\]

\[
S_0^*(n) = h_n(T_0(1), \ldots, T_0(n)).
\]

Hence, \( P_0^*(n) = S_0^*(n) \), which from the definitions seems reasonable enough.

For \( S_0^*(k, n) \) we define the scalar product
On the Number of Isomorphism Classes of Certain Types of Actions and Binary Systems

\[(38) \quad \tilde{a} \cdot \tilde{a} = (e_1, \ldots, e_l) \cdot ((a_{11}, a_{12}), \ldots, (a_{l1}, a_{l2})) = ((\sum e_i a_{ii}) + 1 - t^i, \prod_{i=1}^{l} a_{li}^i), \quad \text{and} \]

\[(39) \quad T(k, n) = \sum_{\tilde{\alpha} = (k, n)} \tilde{\alpha} \times S_{\tilde{\alpha}} \tilde{\alpha}.

For formula (38) compare the definition with formula (33) and the definition preceding formula (11).

Now, letting \(f_{(1,1)}(x_{11}) = x_{11}\) and

\[(40) \quad f_{(k,n)}(x_{11}, \ldots, x_{kn}) = x_{kn} - \sum_{\tilde{\alpha} = (k, n)} \tilde{\alpha} \otimes \tilde{\alpha},

then

\[(41) \quad S_0^*(k, n) = f_{(k,n)}(T(1,1), \ldots, T(k, n)).

From the definition of \(SP\)-primitivity it follows that \(S^*(k, n) = S_0^*(k, n) + S_0^*(k+1, n)\) determines the number of faithful \(SP\)-primitive actions \(Y_X\) such that \(|Y| = n, |X| = k\). Hence, \(S^*(n) = \sum_k S^*(k, n) = 2S_0^*(n) - S_0^*(1, n)\) is the number of faithful \(SP\)-primitive actions \(Y_X\) such that \(|Y| = n\). Now \(S_0^*(1, n) = 1\) if \(n\) is a prime or 1, \(S_0^*(1, n) = 0\) otherwise. Thus \(n\) is a composite number if and only if \(S^*(n)\) is even. A rather inefficient test, to be sure.

\section*{Dropping the faithfulness requirements}

If \(Y_X\) is any action whatsoever, then \(X\) determines a set \(X^* \subseteq Y^X\) by mapping \(x \mapsto f_X : Y \to Y\), where \(f_X y = x y\). Now, if we have a sum \(Y_X + V_U = T_S\), then for \(S^*\) we have a decomposition \(S^* = X^* \cup U^*\) with the proper extension of the definition of the mappings in \(X^*\) and \(U^*\). In other words, if \(Y_X + Y_U = T_S\), then \(Y_{X^*} + V_{U^*} = T_{S^*}\), and hence \(Y_X\) is \(S\)-primitive if and only if \(Y_{X^*}\) is \(S\)-primitive.

If \(p_k(m)\) denotes the number of ordered partitions of \(m\) into \(k\) parts and if \(|X| = m, |X^*| = k\), then there are \(P_k(m)\) ways that \(X\) may generate \(X^*\).

Here \(p_k(m) = \binom{m-1}{k-1}\).

Thus if we let \(K^*(m, n) = \sum_{k=1}^{m} p_k(m) T^*(k, n)\), then \(K^*(m, n)\) denotes the number of \(S\)-primitive actions \(Y_X\) such that \(|Y| = n\) and \(|X| = m\).

If we want to preserve the requirement that the action \(Y_X\) be unitary, then since \(Y_{X^*}\) is also unitary, we merely take the sum \(K_0^*(m, n) = \sum_{k=1}^{m} p_k(m) T_0^*(k, n)\).
to obtain the number of unitary $S$-primitive actions $Y_X$ such that $|Y| = n$ and $|X| = m$.

If we consider an ordinal sum $Y_X + V_U = T_S$, then $S^* = X^* \cup V^* \cup U^*$ in the appropriate way and where $f_\sigma = f_\sigma'$ implies $f_\sigma \cdot y = v = f_\sigma' \cdot y = v'$ and $v = v'$. Thus there is a natural correspondence between $Y_X^* + V_U^*$ and $T_S^*$, so that again $Y_X$ is $OS$-primitive if and only if $Y_X^*$ is $OS$-primitive. Hence, we find that $L^*(m, n) = \sum_{k=1}^n P_k(m)F^*(k, n)$ denotes the number of $OS$-primitive actions $Y_X$ such that $|X| = m$ and $|X| = n$. If we want to restrict ourselves to unitary actions only, then since we have already seen that $Y_X$ is unitary if and only if $Y_X^*$ is unitary, it suffices to take $L_0^*(m, n) = \sum_{k=1}^n P_k(m)F_0^*(k, n)$ to obtain the number of unitary $OS$-primitive actions $Y_X$ such that $|Y| = n$ and $|X| = m$.

For products $Y_XV_U = T_S$, we have $S^* = X^* \times U^*$ in a natural way, i.e., $T_{S^*} = Y_XV_{U^*}$, whence $Y_X$ is $P$-primitive if and only if $Y_X^*$ is $P$-primitive and we have corresponding formulas $M^*(m, n) = \sum_{k=1}^m P_k(m)F^*(k, n)$ and $M_0^*(m, n) = \sum_{k=1}^m P_k(m)F_0^*(k, n)$ to denote the number of $P$-primitive actions $Y_X$ such that $|Y| = m$ and $|X| = n$ and the number of unitary actions $Y_X$ of the same type respectively.

In the case of selective products $Y_XV_U = T_S$ we again obtain an obvious isomorphism between $T_{S^*}$ and $Y_X^*V_{U^*}$, so that by the same arguments as before

\[ N^*(m, n) = \sum_{k=1}^n P_k(m)S^*(k, n) \quad \text{and} \quad N_0^*(m, n) = \sum_{k=1}^n P_k(m)S_0^*(k, n) \]

denote the number of $SP$-primitive actions $Y_X$ such that $|Y| = m$ and $|X| = n$ and the number of unitary actions $Y_X$ of the same type which are also unitary.

**Fixed points**

If $O_1$ denotes the action $1 \cdot 0 = 0$, then if $Y_X$ is an action with an element $0$ such that $x0 = 0$ for all $x \in X$ and such that $y \neq 0$ implies $xy \neq 0$, then we may write $Y_X = Y_X^* + O_1$, where $Y^* = Y - \{0\}$.

Let $T_{0}\alpha(k, n)$ denote the number of faithful unitary actions $Y_X$ without an element of the type $0$, $|Y| = n$, $|X| = k$. If $T_{0\alpha}(k, n)$ denotes the number of faithful unitary actions with precisely $i$ elements of the type $0$, $|Y| = n$, $|X| = k$, then it follows that $T_{0\alpha}(k, n) = T_{0\alpha}(k, n + i)$, so that for $i = 0, 1, 2, \ldots$

\[ (42) \quad T_0(k, n) = T_{0\alpha}(k, n) + T_{0\alpha}(k, n - 1) + \ldots \]
From this we compute \( T_0(k, n) = T_0(k, n) - T_0(k, n-1) \) immediately.

**Binary systems**

Having completed our program for actions, we shall now concern ourselves with binary systems and the sum operations defined for these. We begin with some observations.

1. If \( B = MS(B_1, B_2) \), then \( B = RMS(B_2, B_1) \).
2. If \( B = LS(B_1, B_2) = RS(B_1^*, B_2^*) \), let \( a \in B_1 \cap B_1^* \), \( b \in B_2 \cap B_2^* \). Then \( ab = b = a \), a contradiction since \( B_1 \cap B_2 = \emptyset \). Hence \( B_2 \cap B_2^* = \emptyset \) without loss of generality, i.e., \( B_2^* \subseteq B_1 \), \( B_2 \subseteq B_1^* \). Now let \( a \in B_2^* \), \( b \in B_2 \), then \( ab = b = a \), i.e., \( B_2^* = B_2 \) and \( B_2 \cap B_1 \neq \emptyset \), a contradiction. Hence if \( B = LS(B_1, B_2) \) then \( B \) is RS-primitive.
3. If \( B = RS(B_1, B_2) \) then \( B \) is LS-primitive.
4. If \( B = LS(B_1, B_2) = MS(B_1^*, B_2^*) \), then let \( a \in B_1 \cap B_1^* \), \( b \in B_2 \cap B_2^* \). We have \( ba = b = a \), a contradiction since \( B_1 \cap B_2 = \emptyset \). Hence \( B_2 \cap B_2^* = \emptyset \) without loss of generality, i.e., \( B_2^* \subseteq B_1 \), \( B_2 \subseteq B_1^* \). If \( a \in B_2^* \), \( b \in B_2 \), then \( ab = b = a \), i.e., \( B_2^* = B_2 \) and \( B_2 \cap B_1 \neq \emptyset \), a contradiction. Hence if \( B = LS(B_1, B_2) \) then \( B \) is MS-primitive.
5. If \( B = MS(B_1, B_2) \) then \( B \) is LS-primitive.
6. If \( B = RS(B_1, B_2) \) then \( B \) is MS-primitive and if \( B = MS(B_1, B_2) \), then \( B \) is RS-primitive.
7. The operations are associative. Thus, given \( B_1, B_2, B_3 \) we claim equality in the following situations.
   (a) \( LS(B_1, LS(B_2, B_3)) = LS(LS(B_1, B_2), B_3) \),
   (b) \( RS(B_1, RS(B_2, B_3)) = RS(RS(B_1, B_2), B_3) \),
   (c) \( MS(B_1, MS(B_2, B_3)) = MS(MS(B_1, B_2), B_3) \).

Obviously we have equality of the underlying sets. Now, select \( \{B_i \} \subseteq B_i, i = 1, 2, 3 \) and compute six multiplication matrices \( (b_i b_j) \), \( 1 \leq i, j \leq 3 \) corresponding to the six situations, comparing them pairwise. Thus e.g., the matrix:

\[
\begin{bmatrix}
  e_1^2 & b_2 & b_3 \\
  b_2 & e_2^2 & b_3 \\
  b_3 & b_3^2 & e_3^2
\end{bmatrix}
\]

(43)

corresponds to both sides of (c). Thus it follows that the binary systems on
both sides of (c) are equal.

8. From the associativity of the operations it follows that we have essentially unique decompositions \( B = \text{LS}(B_1, \ldots, B_k) \) with \( B_i \) \( \text{LS} \)-primitive in the case that \( B \) is a finite set, and similarly for the other operations. Thus, if \( B = \text{LS}(B_1, \ldots, B_k) = \text{LS}(C_1, \ldots, C_l) \) are two decompositions of \( B \) into \( \text{LS} \)-primitive subsystems, then \( A_{ij} = B_i \cap C_j \) is either empty or a subsystem. Furthermore, since \( B_1 = \text{LS}(A_{11}, A_{12}, \ldots, A_{1m}) \), where we delete empty sets wherever necessary, it follows that since \( B_1 \) is \( \text{LS} \)-primitive, \( B_1 = A_{i(1)} \) and \( B_1 \subset C_{i(1)} \) for some \( i(1) \). Similarly, \( C_{i(1)} \subset B_{j(i(1))} \) and by the disjointness of the \( B_i, B_1 = C_{i(1)} \) so that without loss of generality we may also take \( B_1 = C_1, B_2 = C_2, \ldots \) etcetera. It follows that \( k = l \) as well.

9. If we consider a finite \( B \), let \( \text{LS} \cdot B \) denote the elements \( \{B_1, \ldots, B_k\} \) occurring in a decomposition \( B = \text{LS}(B_1, \ldots, B_k) \), repeated if necessary and let \( \text{RS} \cdot B, \text{MS} \cdot B \) be defined in a similar fashion. If for a set \( \{B_1, \ldots, B_k\} \) we let \( \text{LS} \cdot \{B_1, \ldots, B_k\} = \{\text{LS} \cdot B_1, \text{LS} \cdot B_2, \ldots, \text{LS} \cdot B_k\} \) (i.e., the union counting repetitions separately) with \( \text{RS} \cdot \{B_1, \ldots, B_k\} \) and \( \text{MS} \cdot \{B_1, \ldots, B_k\} \) similarly defined, then starting with finite \( B \) we obtain a sequence

\[
(44) \quad B \rightarrow \text{LS} \cdot B \rightarrow \text{RS} \cdot (\text{LS} \cdot B) \rightarrow \text{MS} \cdot (\text{RS} \cdot (\text{LS} \cdot B)) \rightarrow \ldots
\]

which must eventually terminate in a set \( \{B_1, \ldots, B_k\} \) of elements \( B_i \) which are primitive, i.e., \( \text{LS} \)-primitive, \( \text{RS} \)-primitive and \( \text{MS} \)-primitive (and thus also \( \text{RMS} \)-primitive as in the introduction, by use of comment 1).

10. From the relations between different kinds of primitivity as described in comments 2, 3, 4, 5 and 6, and the idempotence of the mappings, \( \text{LS} \cdot (\text{LS} \cdot B) = \text{LS} \cdot B \), etcetera, it follows that the primitive parts of \( B \) are also uniquely determined, since in fact the sequence (44) starts for a unique one of the three mappings, i.e., the order of the mappings is essentially immaterial.

11. \( \text{LS}(B_1, B_2) = \text{LS}(B_2, B_1) \) and \( \text{RS}(B_1, B_2) = \text{RS}(B_2, B_1) \) and \( \text{MS}(B_1, B_2) = \text{MS}(B_2, B_1) \) if and only if \( B_1 = B_2 \) for \( \text{MS} \)-primitive binary systems \( B_1 \) and \( B_2 \).

Having noted these facts we are in a position to commence the counting process.

Various numbers

If \( B_L(n) \) denotes the number of \( \text{LS} \)-primitive binary systems \( B \) with \( |B| = n \) then from comment 11 it follows that
On the Number of Isomorphism Classes of Certain Types of Actions and Binary Systems

(45) \[ B(n) = \sum_{\vec{e} \cdot \vec{N} = n} \vec{e} \times B(n), \] and \[ B_L(n) = g_n(B(1), \ldots, B(N)), \]

where the polynomials \( g_n(x_1, \ldots, x_n) \) are those defined as in formula (8).

If \( B_R(n) \) denotes the number of RS-primitive binary systems \( B \) with \(|B| = n\), then from comment 11 it follows that \( B_R(n) = B_L(n) \).

If \( B_M(n) \) denotes the number of MS-primitive binary systems \( B \) with \(|B| = n\), the anticommutative situation applies and

(46) \[ B(n) = \sum_{\vec{e} \cdot \vec{N} = n} \vec{e} \wedge B(n), \] and \[ B_M(n) = G_n(B(1), \ldots, B(n)) \]

where the polynomials \( G_n(x_1, \ldots, x_n) \) are those defined in formulas (19) and (20).

These being the basic quantities required, we shall next consider various other cases.

Suppose \( B_{L,R}(n) \) denotes the number of binary systems \( B \) with \(|B| = n\) which are both LS-primitive and RS-primitive. Then, from comments 2 and 3, it follows readily that

(47) \[ B(n) = B_L(n) + B_R(n) - B_{L,R}(n), \] or \[ B_{L,R}(n) = 2B_L(n) - B(n). \]

Similarly, it follows that

(48) \[ B(n) = B_L(n) + B_M(n) - B_{L,M}(n), \] or \[ B_{L,M}(n) = B_L(n) + B_M(n) - B(n). \]

where \( B_{L,M}(n) \) is the number of binary systems \( B \) with \(|B| = n\) which are both LS-primitive and MS-primitive.

Finally, if \( B_p(n) \) is the number of primitive binary systems \( B \) with \(|B| = n\), then we have equations

(49) \[ B(n) = 2B_L(n) + B_M(n) - 2B_{L,M}(n) - B_{L,R}(n) + B_p(n), \] and

(50) \[ B_p(n) = 2B_L(n) + B_M(n) - 2B(n). \]

If we set \( B_1 \sim B_2 \) if \( B_1 \) and \( B_2 \) have the same primitive parts, and if \( C(n) = \sum_{\vec{e} \cdot \vec{N} = n} \vec{e} \times B(n) \), then \( C(n) \) is the number of equivalence classes \([B_1]\), where \(|B_1| = n\) and \([B_1] = \{B_2 | B_1 \sim B_2 \} \). Also, \( B_p(n) = g_n(C(1), \ldots, C(n)) \).

Another type of decomposition

Suppose \( B \) is a binary system such that \( B = B_1 \cup \cdots \cup B_p \), where \( B_i \subseteq B \), \( B_i \neq \phi \), and where \( B_i \cup B_j \) is one of \( LS(B_i, B_j) \), \( RS(B_i, B_j) \), \( MS(B_i, B_j) \) or \( RMS(B_i, B_j) \). Then \( \{B_1, \ldots, B_p\} \) is a decomposition of \( B \). Also, \( B \) is indecomposable if the only decomposition of \( B \) is \( \{B\} \). Notice that if \( B \) is indecomposable, then it is primitive.
since $B = \text{LS}(B_1, B_2)$ implies $\{B_1, B_2\}$ is a decomposition. Suppose $\{B_1, \ldots, B_k\}$ and $\{C_1, \ldots, C_i\}$ are decompositions of $B$. Let $A_{ij} = B_i \cap C_j$, then if $A_{ij} \neq \emptyset$, we certainly have $A_{ij} \subseteq A_{ij}$. Suppose we consider $A_{ij}$ and $A_{rs}$, with $(i, j) \neq (r, s)$. Then, $i \neq r$ without loss of generality, and $B_i \cup B_r = \text{LS}(B_i, B_r)$ without loss of generality. It follows that then also $\text{LS}(A_{ij}, A_{rs})$, so that if $A = \{A_{ij} | A_{ij} \neq \emptyset\}$, then $A = \{B_1, \ldots, B_k\}$ \cap $\{C_1, \ldots, C_i\}$ is also a decomposition of $B$. In particular, if $B$ is finite, then the intersection of all decompositions of $B$ yields a unique finest decomposition $\{B_1, \ldots, B_k\}$ whose elements are themselves indecomposable. Our next object is to say something about the number $B_i(n)$ of indecomposable binary systems $B$ such that $|B| = n$.

We shall begin by counting those binary systems $B$ with the property that $|B| = k$ and $\{B_1, \ldots, B_k\}$ is a decomposition with $B_i = \{b_i\}$ the singleton system $b_i'. These systems are quite obviously characterized by the rule $b_i' \subseteq \{b_i, b_j\}$. In terms of a colouration associated with $B$, we consider the set $(k \times k)_0 = \{(i, j) \mid i \neq j\}$, and we colour the elements $(i, j)$ by 0 or 1 according to the rules $X(i, j) = 1$ if $b_i' = b_j'$ and $X(i, j) = 0$ if $b_i' = b_i'$. Conversely, any colouration immediately determines a binary system of the type. We operate with $S_k$ on $(k \times k)_0$ by $\bar{\phi}(i, j) = (\phi(i), \phi(j))$ for $\phi \in S_k$. Then it follows that if $\phi$ is of type $1^{\mu_1} \cdots s^{\mu_s}$, then $\bar{\phi}$ is of type $1^{\mu_1} \cdots s^{\mu_s}$ where $t = \max\{i, j \mid 1 \leq i, j \leq s\}$ and $\mu_d = \mu_d^{(2)} - \mu_d$, with $\mu_d^{(2)} = \sum_{i, j = d} (i, j) \mu_i \mu_j$, and $\mu_d = 0$ if $d > s$. It follows that the cycle index is

$$
\sum_{k = \mu_1 + 2\mu_2 + \cdots + s\mu_s} (\mu_1! \cdot 2^{\mu_2} \cdots s^{\mu_s})^{-1} x_1^{\mu_1} \cdots x_t^{\mu_t},
$$

and the number in question is

$$
\sum_{k = \mu_1 + 2\mu_2 + \cdots + s\mu_s} (\mu_1! \cdot 2^{\mu_2} \cdots s^{\mu_s})^{-1} x_1^{\mu_1} \cdots x_t^{\mu_t}.
$$

Of course if we let this number be $D_k$, then the number of binary systems $B$ with finest decomposition $\{B_1, \ldots, B_k\}$ and $B_1 \neq B_2 = B_k$ where $|B| = n$ is given by:

$$
\sum_{k \mid n} D_k B_i(n/k).
$$

In order to handle the general situation, suppose that $D_k = D\{k\} = D(e_1, \ldots, e_t)$ is the number of binary systems with the decomposition $\{B_1, \ldots, B_1, \ldots, B_n, \ldots, B_t\}$ ($e_t$ copies of $B_t, B_1, \ldots, B_1$ distinct). We let the group $S_e \times \cdots \times S_e$, act on $((e_1 + \cdots + e_t) \times (e_1 + \cdots + e_t))_0$ via $\phi_1 \cdots \phi_t(j) = \phi_1(j - (e_1 + \cdots + e_{i-1})) + (e_1 + \cdots + e_{i-1})$ if $e_1 + \cdots + e_{i-1} < j \leq e_1 + \cdots + e_{i-1}$.
On the Number of Isomorphism Classes of Certain Types of Actions and Binary Systems: 175

+e_j, and if \( \phi=(\phi_1, \ldots, \phi_t) \), \( \phi(i,j)=(\phi(i), \phi(j)) \). If \( \phi_i \) has the type \( 1^{\mu_i} \cdots s_1^{\mu_i}, \mu_{ij} \geq 0 \), it follows that \( \phi=(\phi_1, \ldots, \phi_t) \) has type \( 1^{\mu_1} \cdots s_1^{\mu_1} \) where \( \mu_j=\sum \mu_{ij} \) and \( \phi \) has the type described above formula (51).

The number of elements of \( S_1 \times \cdots \times S_t \) conjugate to \( \phi \) is

\[
\prod_{i=1}^t [(\mu_{i1}! 2^\mu_{i1} \cdots S_1^{\mu_{i1}})^{-1} e_i!] = \alpha(\mu_{ij})
\]

If we number the parts \( B_1, \ldots, B_t \) continuously \( B_1, \ldots, B_{e_1}, \ldots, B_t \), and if we set \( X(i, j)=1 \) if \( b_i=b_j \) and \( X(i, j)=0 \) if \( b_i=b_j \), then we have a colouration, and conversely every colouration determines a binary system. Now, isomorphisms move the parts of the decomposition around according to mappings \( \phi \), and thus if \( P(S_1 \times \cdots \times S_t) \) denotes the cycle index, it follows that \( D_\phi=P(S_1 \times \cdots \times S_t; 2, \ldots, 2) \).

Note that the cycle index is given by

\[
\sum_{e_1=1^{\mu_1}+\cdots+s_1^{\mu_1}} \alpha(\mu_{ij})/e_1! \cdots e_t! x_1^{\mu_1} \cdots x_t^{\mu_t}
\]

where \( \nu_d = \mu_d^{(2)} - \mu_d \) as described above. For example if \( e_1=\cdots=e_t=1 \), then \( D_\phi=2^{n-t} \).

Suppose that \( \vec{e}=(e_1, \ldots, e_1, \ldots, e_t, \ldots, e_t) \) where \( e_1 \gg \cdots \gg e_t \) and there are \( s_i \) copies of \( e_i \). Let \( \vec{N}=(N_1, \ldots, N_1, \ldots, N_t, \ldots, N_t) \) where \( N_1 \geq N_2 \geq \cdots \geq N_t \). Write \( B_1(\vec{N})=(B_1(N_1), \ldots, B_t(N_1, s_1, \ldots, N_1)) \) and \( M^* i = k \) if the number \( M \) appears \( k \) times in the vector \( B_1(\vec{N})=(B_1(N_1), \ldots, B_t(N_1, s_1, \ldots, N_t)) \). Furthermore take

\[
M* B_1(\vec{N}) = \left( \sum_{i=1}^t M^{* i} \right)
\]

Finally, set \( [B_1(\vec{N})] = \prod_{M=1}^\alpha M* B_1(\vec{N}) \) where \( \alpha \) is sufficiently large (since then \( \alpha* B_1(\vec{N})=1 \)).

Then, we claim that:

\[
B(n) = \sum_{\vec{N}=n} [B_1(\vec{N})] D_\phi
\]

Since \( D_{(1)}=1 \), it follows that

\[
B_1(n) = B(n) - \sum_{\vec{N}=n} [B_1(\vec{N})] D_\phi
\]

which can be computed recursively. It only remains to verify formula (57). What we do is the following. Given any binary system \( B \) with \( |B|=n \) it has a cover \( \{B_1, \ldots, B_1, \ldots, B_t, \ldots, B_t\} \) where there are \( e_i \) copies of \( B_i \). We may sort these
according to number \((e_1 \geq e_2 \geq \cdots \geq e_t)\) and within a fixed number of copies \(e_i\) of which there are several we order according to size \((N_{1t} \geq N_{2t} \geq \cdots \geq N_{st})\), i.e., we are working with the usual cascading vectors. In effect, if we apply this system we have used up all available freedom and the method now follows the usual rules. Thus \(\tilde{e} \cdot \tilde{N} = n\), since we are taking the union of subsets. Also, in each slot we have a certain number of possible choices, viz., \(B_i(N_{ij})\) in the slot corresponding to \(N_{ij}\). Next, if we have \(M^{*i}\) components in the decomposition which occur with the same frequency and which have the same number of elements, then we have \((M^{*i})!\) equivalent arrangements. Thus, since the total number of times we have to select different components with \(M\) elements is \(\sum_{i=1}^{t} M^{*i}\), the total contribution to the possibilities for the type of arrangement we have described consists of the product of the two coefficients

\[
\left(\frac{B_i(M)}{\sum_{i=1}^{t} M^{*i}}\right) \left(\sum_{i=1}^{t} M^{*i}\right) = M^{*i}B_i(N).
\]

Indeed, the first coefficient counts the number of selections, while the second coefficient counts the number of arrangements. Finally, since binary systems with different numbers of elements are sowieso not isomorphic, it follows that with the vectors \(\tilde{e}\) and \(\tilde{N}\) fixed, that the total count of possibilities is simply the product. Thus \(B_i(N)\) counts the number of distinct decompositions, which then needs to be multiplied by \(D_{\tilde{e}}\) to give the number of ways they can be put together. Summing over all proper bipartitions yields formula (57) and we're done.

The University of Alabama
University, Alabama 35486

REFERENCES