ON THE RICCI TENSORS OF PARTICULAR FINSLER SPACES

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In the present paper we shall study various Ricci tensors in particular Finsler spaces. The problem of the Ricci tensors of Finsler spaces is stated, for example, in the paper [15] § 2 and will be important in applications of Finsler geometry to the theoretical physics. One of the difficulties of the problem is that the Ricci tensors defined from the \( h \)-curvature tensor \( R_{hijk} \) and \( hv \)-curvature tensor \( P_{hijk} \) are not symmetric in general, contrary to the case of Riemannian geometry. This study was promoted by Professor Y. Takano's report [25], continuing [15] § 2.

In § 1 two kinds of the \( hv \)-Ricci tensors, denoted by \( P_{ij}^{(1)} \) and \( P_{ij}^{(2)} \), and the \( h \)-Ricci tensor \( R_{ij} \) are introduced. The purpose of the next section is to consider the Bianchi identities and to produce various identities related to the Ricci tensors. The so-called conservation law is important in the physics. We find a tensor field which satisfies the law under some assumption (Theorem 1). The third section is devoted to studying the Ricci tensors of a \( C \)-reducible Finsler space, which is defined by Prof. M. Matsumoto [7] and will be important in the physics. In § 4 we shall touch upon isotropic Finsler spaces due to H. Akbar-Zadeh [1]. We shall treat, in § 5, Finsler spaces of scalar curvature owing to L. Berwald [2]. It is shown that in such a space the \( h \)-Ricci tensor \( R_{ij} \) is symmetric and the condition \( R_{ij} = \nu k_{ij} \) yields "of constant curvature" (Theorem 10). Furthermore we pay attention to Finsler spaces of scalar curvature which satisfy the \( C \)-reducibility. The final section is a list of various known results of the \( v \)-curvature tensor \( S_{hijk} \) and \( v \)-Ricci tensor \( S_{ij} \).

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§ 1. The \( hv \)-and \( h \)-Ricci tensors.

Let \( F^* \) be an \( n \)-dimensional Finsler space with the fundamental function
We denote by $g_{ij}(x,y) = (\partial^2 L^2/\partial y^i \partial y^j)/2$ the fundamental tensor. The angular metric tensor $h_{ij}$ is given as $h_{ij} = g_{ij} - l_{ij}$, where $l_{ij} = \partial L/\partial y^i$ is the normalized element of support. Hereafter the terminologies and notations are the same as in the monograph [12].

The $h^v$-curvature tensor $P_{hijk}$ (cf. (2.3)) satisfies the four identities ([12] §17):

\[ P_{hijk} = -P_{thjk}, \]
\[ P_{hiok} = P_{hiko} = 0, \]
\[ S_{(ki)j} \{ P_{hijk} \} = 0, \]
\[ S_{(hi)k} \{ P_{hijk} \} = 0, \]

where $S_{(kij)}$ means cyclic permutation of indices $h, i, j$ and summation and the index $o$ means contraction by the element of support $y^i$.

By virtue of (1.1), (1.3) and (1.4) the number $N$ of the independent components of the $h^v$-curvature tensor $P_{hijk}$ is given in the formula

\[ N = n^2 (n-1)(n+4)/6. \]

From this we have, for example, $N = 4$ ($n = 2$), $N = 21$ ($n = 3$), $N = 64$ ($n = 4$) and so on.

Next we define the $h^v$-Ricci tensors from the $h^v$-curvature tensor $P_{hijk}$ in the following forms:

\[ P_{ij}^{(1)} = P_{i^s^j}, \quad P_{ij}^{(2)} = P_{i^t^j}, \]

where $P_{i^s^j} = g^{ms} P_{imsj}$ and $P_{i^t^j} = g^{ms} P_{imjs}$.

As a matter of course $P_{ij}^{(1)} \neq P_{ij}^{(2)}$ in general, but we can see that the skew-symmetric parts of $P_{ij}^{(1)}$ and $P_{ij}^{(2)}$ are equal to each other, which was suggested by Prof. M. Matsumoto.

**Proposition 1.** The skew-symmetric parts of the two $h^v$-Ricci tensors $P_{ij}^{(1)}$ and $P_{ij}^{(2)}$ are equal:

\[ P_{ij}^{(1)} - P_{ji}^{(1)} = P_{ij}^{(2)} - P_{ji}^{(2)} = P_{ij}^s, \]

where $P_{ij}^s = g^{sm} P_{ijm}$.

**Proof.** Multiplying (1.3) by $g^{mi}$ and summing over $m$ and $k$ we have

\[ P_{hj}^{(2)} - P_{jk}^{(2)} = P_{hj}^m. \]

Similarly from (1.4) we get

\[ P_{ik}^{(1)} - P_{ki}^{(1)} = P_{ik}^m. \]

Consequently we have proved the identities as above.

**Remark 1.** A Finsler space with $P_{hijk} = P_{hikj}$ is called $P$-symmetric ([13])
If a Finsler space is $P$-symmetric, we have the unique $h\nu$-Ricci tensor $P_{ij} = P_{ij}^{(1)} = P_{ij}^{(2)}$; this is a very convenient fact, but, contrary to our expectation, the scalar curvature $P = P_{ij}g^{ij}$ necessarily vanishes in this case ([15] § 2).

**Remark 2.** A Finsler space with the Cartan connecton $C^\nu$ is $P$-symmetric, if and only if the $\nu$-curvature tensor $S_{hi,jk}$ satisfies the equation ([12] (17.25), [15] Prop. 2)

$$S_{hi,jk} = 0$$

where $(\cdot)$ denotes the $h$-covariant differentiation with respect to $C^\nu$.

We turn the consideration to the $h$-curvature tensor $R_{ij}^{\prime, jk}$ which is given in the form

$$(1.5) \quad R_{ij}^{\prime, jk} = A_{(ijk)} \{ \partial_k F_i^{\prime, j} + F_i^{\prime, j} F_i^{\prime, k} \} + C_{ij}^{\prime} R_{jk}^{\prime},$$

where $A_{(ijk)}$ means interchange of indices $j, k$ and subtraction and $\partial_k = \partial_k - N_{ij}^{\prime} \partial_l$.

If we adopt the Cartan connection $C^\nu$ as the Finsler connection, the following identities hold ([12] (17.9), (17.10), (22.7), (22.8)):

$$(1.6) \quad R_{ij}^{\prime, jk} = - R_{ij}^{\prime, k} = - R_{ij}^{\prime, k},$$

$$(1.7) \quad S_{(ijk)} \{ R_{hi,jk} \} = - S_{(ijk)} \{ C_{ij}^{\prime} R_{r, ij} \},$$

$$(1.8) \quad R_{j, khi} = R_{j, khi} + N_{hi, jk},$$

where $N_{hi, jk} = A_{(ijk)} \{ C_{ij}^{\prime} R_{r, hjk} - C_{ij}^{\prime} R_{r, ik} \}$.

**Remark 3.** In a Finsler space of scalar curvature, the $h$-curvature tensor $R_{hi, jk}$ satisfies $S_{(ijk)} \{ R_{hi, jk} \} = 0$ and $R_{j, khi} = R_{j, khi}$ as Riemannian curvature tensor. See § 5.

We define the $h$-Ricci tensor $R_{ij}$ from the $h$-curvature tensor $R_{hi, jk}$ in the form

$$R_{ij} = R_{ij}^{\prime, js}.$$  

The $h$-Ricci tensor $R_{ij}$ is also equal to $R_{ij}^{\prime, sj}$ because of $R_{hi, jk} = R_{ih, jk}$.

In case of the Cartan connection $C^\nu$ it is observed that

**Proposition 2.** The skew-symmetric part of the $h$-Ricci tensor $R_{ik}$ is given by the equation

$$(1.9) \quad R_{ik} - R_{ki} = C_i R_{ik}^{\prime} + C_{ij}^{\prime} R_{ik}^{\prime} - C_{ij}^{\prime} R_{ik}, \quad (C_i = C_{ij, mn}).$$

**Proof.** From (1.5) we obtain easily

$$R_{ik} - R_{ki} = \partial_k F_i^{\prime, j} - \partial_j F_i^{\prime, k} + C_{ij}^{\prime} R_{ik}^{\prime} - C_{ij}^{\prime} R_{ik}.$$
where $F^{i}_{h} = \gamma^{i}_{h} - C_{h}N^{i}_{h} + \delta_{h} \log \sqrt{g}$ ([12] (17.3)'). Applying $\partial_{k}$ to the above, we get

$$\partial_{k}F^{i}_{h} = \partial_{h}F^{i}_{k} = C_{i}R^{i}_{kh}.$$ 

Thus the proof is completed.

It is noted that the contraction of (1.7) by $g^{ij}$ yields immediately another proof of (1.9).

§ 2. The Bianchi identities and Ricci tensors.

In the theory of general Finsler connections, devoted in the monograph [12], we have four Jacobi identities in combination with two vector fields, called the $h$- and $v$- basic vector fields. From each Jacobi identity, we obtain three identities, which show the vanishing of the $h$- horizontal part, $v$- horizontal part and vertical part of the Jacobi identity. Hence there are twelve identities. Because one of these identities is trivial, we have finally eleven Bianchi identities which are classified into four groups ([12] § 11).

We are specially concerned with the Cartan connection $C^\prime$. The four groups of the Bianchi identities of $C^\prime$ are as follows ([12] § 17):

**The first group**

(BC-I-1) \[ S_{ijbk} \{ C_{r}^{i}R^{r}_{jk} - R_{r}^{i}jk \} = 0, \]

(BC-I-2) \[ S_{ijbk} \{ P_{ir}^{h}R_{jk} + R_{k}^{h}ij \} = 0, \]

(BC-I-3) \[ S_{ijbk} \{ P_{mr}^{h}R_{jk} + R_{m}^{h}kij \} = 0. \]

**The second group**

(BC-II-1) \[ A_{ij} \{ C_{k}^{i}k + C_{r}^{i}P_{r}^{j}jk - P_{r}^{j}jk \} = 0, \]

(BC-II-2) \[ R^{h}_{ij} + A_{ij} \{ R_{ir}^{h}C^{j}k + P_{ir}^{h}P_{r}^{j}jk + P_{r}^{j}kij \} = 0, \]

(BC-II-3) \[ R_{m}^{h}k_{ij} + S_{m}^{h}k_{r}R_{r}^{i}j + A_{ij} \{ R_{m}^{h}C^{j}k + P_{m}^{h}P_{r}^{j}jk + P_{m}^{h}k_{ij} \} = 0, \]

where (1) denotes the $v$- covariant differentiation with respect to $C^\prime$.

**The third group**

(BC-III-1) \[ A_{ij} \{ C_{r}^{h}j - C_{j}^{h}P_{r}^{i}i \} - S_{r}^{h}jk = 0, \]

(BC-III-2) \[ A_{ij} \{ P_{ir}^{h}C_{r}^{j}j - P_{r}^{h}kj + P_{r}^{h}kj \} = 0, \]

(BC-III-3) \[ S_{m}^{h}k_{ij} + A_{ij} \{ P_{m}^{h}rC_{r}^{j}j - S_{m}^{h}k_{r}P_{r}^{j}kj - P_{m}^{h}k_{ij} \} = 0. \]

**The fourth group**

(BC-IV-1) \[ S_{ij} \{ S_{k}^{h}jk \} = 0, \]

(BC-IV-2) \[ S_{ij} \{ S_{m}^{h}ij \} = 0. \]
It is remarked that (BC-I-2), (BC-II-2) and (BC-III-2) is a consequence of (BC-I-3), (BC-II-3) and (BC-III-3) respectively.

From (BC-I-1), contracting for \( h \) and \( j \) we have (1.9) because (BC-I-1) is quite the same with (1.7).

The contraction of (1.9) by \( y^h \) yields

\[(2.1) \quad R_{ko} - R_{ok} = C_i R_{ki}^i + C_{k}^{i} r R_{o}^{i}.
\]

By virtue of the metrical property of the Cartan connection \( C^r \), a contraction and \( h- \) (or \( v- \)) covariant differentiation are commutative.

Next we contract (BC-I-3) for \( h \) and \( i \) to obtain

\[(2.2) \quad R_{mk} = R_{mi} - P_m^{(1)} R_{ki} = R_{mk}^h + P_{mh}^h R_{rk}^k - P_{mr}^h R_{rik}.
\]

In Riemannian geometry (2.2) yields the important equation \( R_{i} = 2 R_{ir} \) by contracting by \( g^{mk} \) ([22] p.18).

Contracting (2.2) by \( y^m \) and \( y^m y_k \), we have respectively

\[(2.2)' \quad R_{ok} - R_{oi} = P_{or}^{(1)} R_{ki} = R_{mk}^h + P_{mh}^h R_{rk}^k - P_{mr}^h R_{rik}.
\]

\[(2.2)'' \quad R_{op} = R_{oi} - P_{or}^{(1)} R_{ki} = R_{mk}^h + P_{mh}^h R_{rk}^k - P_{mr}^h R_{rik}.
\]

Next the Bianchi identity (BC-II-1) is, by virtue of (1.1) and (1.3), rewritten as

\[(2.3) \quad P_{hijk} = C_{ijk}^h - C_{hjk}^i + P_{ikr} C_{r}^h - P_{hrk} C_{r}^i,
\]

which is nothing but the well-known representation of the \( h-v- \) curvature tensor \( P_{hijk} \) of the Cartan connection \( C^r \) ([12] (17.23)).

From (2.3) the \( h-v- \) Ricci tensors \( P_{hk}^{(1)} \) and \( P_{hk}^{(2)} \) are written as follows:

\[(2.4) \quad P_{hk}^{(1)} = C_{k}^r - C_{h}^{rs} = P_{k}^{rs} C_{r}^h - P_{hrk} C_{r}^i, \quad (P_{k}^{rs} = g^{sm} P_{kmr}).
\]

\[P_{hk}^{(2)} = C_{k}^r - C_{h}^{rs} = C_{r}^h - P_{hrk} C_{r}^i.
\]

Consequently it is easily observed that

**Proposition 3.** The \( h-v- \) Ricci tensors \( P_{hk}^{(1)} \) and \( P_{hk}^{(2)} \) of the Cartan connection \( C^r \) satisfy the following equations:

\[(1) \quad P_{ok}^{(1)} = P_{ok}^{(2)} = C_{k}^i, \quad P_{ho}^{(1)} = P_{ho}^{(2)} = 0,
\]

\[(2) \quad P_{hk}^{(1)} - P_{hk}^{(2)} = A_{(hk)} \{ C_{k}^i h + P_{hrk} C_{r}^i \},
\]

\[(3) \quad P_{hk}^{(1)} - P_{hk}^{(2)} = P_{hrk} C_{r}^i + P_{h}^{rs} C_{r}^k - P_{hrk} C_{r}^i - C_{hrk} C_{r}^i.
\]

**Remark 4.** According to Remark 1 the \( P- \) symmetry implies that the right hand side of (3) in Proposition 3 vanishes but the inverse does not hold. Further the \( h-v- \) Ricci tensor \( P_{ij} \) is not symmetric even if \( P- \) symmetry holds good.
From (BC-II-3) we take the contraction for $h$ and $j$ to get
\[(2.5)\]
\[R_{tk}i + R_{tr}C_{t}^{i} + P_{it}^{(1)}i = P_{ik}^{(1)}\]
\[= P_{tk}i|_{r} - R_{tk}C_{t}^{i} - P_{tk}iP_{r}r + S_{tr}R_{tk}R_{tr}.\]
Contracting (2.5) by $y^{l}$ and $y^{k}$, in view of $P_{oi}^{(1)} = C_{i}^{o}o$ we get respectively
\[(2.5)^{\prime}\]
\[R_{th}i - R_{ik} + R_{or}C_{t}^{i} + C_{i}^{o}o = P_{k}rR_{r}i + P_{o}rP_{t}r + S_{tr}R_{tk}R_{tr}.\]
\[(2.5)^{\prime\prime}\]
\[R_{oo}i + C_{i}^{o}oi = R_{oi} + R_{io} - R_{or}C_{t}^{i}i\]
\[= 2R_{oi} + C_{r}R_{r}ic.\]

In the last equation we refered to (2.1).

In (BC-II-3), we take another contraction for $h$ and $i$ to get
\[(2.6)\]
\[P_{ij}^{(2)}i + P_{ik}^{(2)}j + S_{tr}R_{tk}^{j} = R_{tk}^{j}i + A_{j}^{(j)}R_{tk}^{i}C_{t}^{i} + P_{tk}^{r}P_{r}r,\]
where $S_{tr}$ is the $\nu$-Ricci tensor $S_{tr} = S_{tr}^{r}r$.
Contract (2.6) by $y^{l}$ and $y^{k}$. Paying attention to $y^{l}i = 0$ and $y^{l}j = \delta^{l}j$, in virtue of (1) of Prop. 3 we get
\[(2.6)^{\prime}\]
\[C_{j}^{i}0 + A_{k}^{0}j = R_{tk}^{j}i + R_{tk}^{j} - R_{tk}^{j}C_{t}^{j}i,\]
\[(2.6)^{\prime\prime}\]
\[C_{j}^{i}0 = R_{tk}^{j}i - R_{ij} - R_{or}C_{t}^{j}i.\]

We compare (2.6) with (2.5) : Eliminating the term $C_{i}^{o}oi$ from them we obtain
\[(2.7)\]
\[R_{oo}i + R_{ik}^{k}i - 2R_{oi} - R_{io} = 0.\]
In Riemannian geometry (2.7) reduces to a trivial equation.

Next from (BC-III-2), in virtue of (1.1) and (1.4) another expression of the $\nu\nu$-curvature tensor $P_{hij}$ is obtained ([12][17.27]):
\[(2.8)\]
\[P_{hij} = P_{ij} + P_{ik}C_{j}^{i} - P_{ik}i + P_{tk}C_{j}^{i} - P_{hk}C_{j}^{i}.\]
The contraction of (2.8) by $g^{ij}$ and $g^{ik}$ yields respectively
\[(2.4)^{\prime}\]
\[P_{ik}^{(1)} = C_{i}^{i}oH - P_{ik}o + P_{k}tC_{t}^{i} - P_{k}tC_{t}^{i},\]
\[(2.4)^{\prime\prime}\]
\[P_{ik}^{(2)} = C_{i}^{i}oH - P_{ik}o + C_{t}^{i}tC_{t}^{i} - P_{k}tC_{t}^{i} - P_{k}tC_{t}^{i}.\]
This is another representation of (2.4). We make a comparison with (2.4) and (2.4) to get the following interesting equation:
\[(2.9)\]
\[C_{i}^{i}oH - C_{i}^{i}oH = P_{t}rj - C_{t}^{i}r.\]
Hence we conclude
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**Proposition 4.** In a Finsler space $F^n$ with the Cartan connection, the tensor

$$C_{i|\theta}j - C_{i|j}$$

is written as (2.9) and symmetric in $i$ and $j$.

From (2.4)' it is observed that the equation (2) of Proposition 3 is written in the form

$$P_{hk}^{(1)} - P_{kh}^{(1)} = P_{hk}^{(2)} - P_{kh}^{(2)} = A_{(hk)} \left[ C_{k|\theta}j - C_{j|\theta}k + P_{i}^{j}r C_{r}^{i} \right].$$

Now we are concerned with (BC-III-3). The contraction for $h$ and $k$ yields

(2.10) $$P_{mj}^{(1)} |i - P_{mi}^{(1)} |j = S_{m|ij} + A_{(ij)} \left[ P_{m}^{i}r C_{r}^{i}j + S_{m}^{r}r P_{rj} \right],$$

which corresponds to the equation (2.6). The contraction of (2.10) by $g^m$ gives

(2.10)' $$P_{j|\theta}i - P_{i|\theta}j = P_{ij}^{(1)} - P_{ji}^{(1)} + P_{i}^{r}r C_{r}^{i}j - P_{j}^{r}r C_{r}^{i}i.$$

Substituting from (2) of Prop. 3 into the above, we get an interesting equation

(2.10)'' $$P_{j|\theta}i - P_{i|\theta}j = C_{j}^{i} - C_{i}^{j}.$$

It is, however, remarked that (2.10)'' is solely a consequence of Proposition 4.

In (BC-III-3), we take another contraction for $h$ and $j$ to obtain

(2.11) $$P_{mk}^{(2)} |i + P_{mr}^{(2)} C_{k}^{i} = S_{m|ik} + S_{mr} P_{k}^{i}r,$$

Lastly we are concerned with the Bianchi identities of the fourth group (BC-IV-1, 2). From (BC-III-1), as $C_{hji|k} = C_{hik} |j$ in the Cartan connection $C'I$, it follows that

$$S_{hijk} = A_{(jk)} \left[ C_{r}^{k}C_{rij} \right],$$

which is nothing but the well-known representation of the $v$-curvature tensor $S_{hijk}$ of the Cartan connection $C'I$ ([12](17.20), See §6). Substituting the above into (BC-IV-1, 2), these are automatically satisfied ([12] §7).

Here we shall return to (2.2)''. Because of $P_{or}^{(1)} = C_{r|\theta}$, if we put

(2.12) $$Z_{i} = P_{i}^{r}r R_{so}^{r} - R_{so}^{r} C_{r|\theta},$$

then (2.2)'' is written in the form

$$R_{oo} |i - R_{oi} |o - R_{oi} |r = Z_{i},$$

so that we conclude
THEOREM 1. In a Finsler space with $Z_i=0$ the tensor $B^i_j = R^i_{k0} + R^j_{00} \delta^i_j - R^j_{0i} \gamma^k$ satisfies the conservation law $2) B^h_{i|k} = 0.$

REMARK 5. Although the assumption $Z_i=0$ seems not to be natural, this is identically satisfied in a Finsler space of scalar curvature, as will be proved in §5 Th. 12. In a Riemannian space the above conservation law $B^h_{i|k} = 0$ is a consequence of $R_{jki} = R_{ijk} - R_{ikj}.$

Next we shall be concerned with the conservation law with respect to the $v-$covariant differentiation. (See §6 as to the $v-$curvature tensor $S_{hijk}$). The equation (2.7) is notable in this point of view. Suppose that $2R_{oi} + R_{io} = 0.$ Then we get $R_{oo} = 0$ and (2.7) reduces to $R_{io} | h = 0.$ Consequently

THEOREM 2. In a Finsler space with $B_i = 2R_{oi} + R_{io} = 0$ the contracted \((v)\) $h-$torsion tensor $R^h_{io}$ satisfies the conservation law $R^h_{io} | h = 0.$

REMARK 6. In a Finsler space of scalar curvature we have $R_{oi} = R_{io}$ (See §5 Th. 9(3)). Hence the tensor $B_i$ in Theorem 2 is written in the form $B_i = 3R_{io} = (n-2)L^2 K_{il} + 3(n-1) K L l_i.$

Contrasting the above by $\gamma^i$ we get $B_o = 3(n-1)K L^2.$ Consequently $B_i = 0$ is equivalent to $K = 0$ and Theorem 2 is trivial.

Consider the above condition $B_i = 0.$ It follows from (2.1) that the contracted Ricci tensors $R_{oi}$ and $R_{io}$ are expressible in

$$R_{oi} = - R_{io}/2 = -(R_{tio} + C^r_{tr} R_{ro}^r)/3, \quad (R_{tio} = C^m R_{mot}).$$

From the above result and (2.6)" we have

COROLLARY. In a Finsler space, where the contracted Ricci tensors $R_{oi}$ and $R_{io}$ are written in the form (2.13), the contracted \((v)\) $h-$torsion tensor $R^h_{io}$ satisfies the conservation law $R^h_{io} | h = 0$ and $C_{i|l|o|o} = (R_{tio} - 2C^r_{tr} R_{ro}^r)/3.$

§3. $C-$reducible Finsler spaces and Ricci tensors.

In the present section we are concerned with $C-$reducible Finsler spaces. Because, for instance, the Randers space (See §6), which is important in the theoretical physics, is certainly $C-$reducible ([12] §36, [19], [27]).

DEFINITION. A non-Riemannian Finsler space $F^n (n \geq 3)$ is called $C-$reducible ([7]), if the $h-$torsion tensor $C_{ijk}$ is written in the form

$$C_{ijk} = (C_{ihjk} + C_{jhki} + C_{hij})/(n+1).$$

It is well known that the $v-$curvature tensor $S_{ijk}$, the $(v)h-$torsion tensor

2) See [12] §26, [18].
$P_{ijk}$ and the $hv$-curvature tensor $P_{hk}$ of $F^n$ are respectively written in the following concrete form ([12] §30):

\begin{align}
S_{ijk} &= A_{ij} \{ h_{ik}C_{jk} + h_{jk}C_{ik} \} / (n+1)^2, \\
P_{ijk} &= G_{ijk} + G_{jki} + G_{kij}, \\
\rho_{hijk} &= N_{hi}h_{jk} + A_{(i)}[h_{ij}N_{hk} + h_{ik}N_{jh}],
\end{align}

where $h_{ij}$ is the angular metric tensor and

\begin{align}
C_{ij} &= C^2h_{ij} / 2 + C_iC_j, \quad (C^2 = C_iC^i),
G_i &= C_{i \cdot o} / (n+1), \\
N_{ij}^{(1)} &= (C_{i \cdot j} - C_{i}G_{j} - \mu h_{ij} / 2) / (n + 1), \quad (\mu = C_{i}G_{i}), \\
N_{ij}^{(2)} &= (C_{i \cdot j} + C_{j}G_{i} + \mu h_{ij} / 2) / (n + 1), \\
N_{ij} &= -N_{ij}^{(1)} + N_{ij}^{(2)} = -N_{ij}^{(2)} + N_{ij}^{(2)}.
\end{align}

**Remark 7.** From (3.3) it is observed that the $C$-reducible Finsler space is $P$-reducible ([13]). The converse does not hold good in general. Recently C. Shibata has showed ([21]) that in case of a Finsler space of scalar curvature the converse is correct.

From (3.4) the $hv$-Ricci tensors $P_{hk}^{(1)}$ and $P_{hk}^{(2)}$ are especially written as follows:

\begin{align}
P_{hk}^{(1)} &= \{ nC_{k}h_{i \cdot k} - C_{k}h_{i \cdot k} \} + L^{-1}(n + 1)[G_{k}h_{i \cdot k} + G_{i}h_{k}] + 2C_{k}G_{k} \\
&\quad - (n - 1)C_{k}G_{k} - \varepsilon h_{k} / (n + 1), \quad (\varepsilon = C_{i} + (n - 1) \mu), \\
(3.5)
P_{hk}^{(2)} &= \{ nC_{k}h_{i \cdot k} - C_{k}h_{i \cdot k} \} + L^{-1}(n + 1)[G_{k}h_{i \cdot k} + G_{i}h_{k}] - 2C_{k}G_{k} \\
&\quad + (n - 1)C_{k}G_{k} - (\varepsilon - 2(n - 1) \mu) h_{k} / (n + 1).
\end{align}

From (3.5) the $hv$-scalar curvatures $P^{(1)} (= P_{hk}^{(1)} g^{kk})$ and $P^{(2)} (= P_{hk}^{(2)} g^{kk})$ are written in the form

\begin{align}
P^{(1)} &= -P^{(2)} = - (n - 2) \mu.
\end{align}

Suppose that $P^{(1)} = 0$, i.e., $\mu = C_{i}G_{i} = 0$. As $P_{ijk} = C_{i \cdot j \cdot k \cdot o}$, it is remarked that $G_{i} = P_{i} / (n + 1)$, where $P_{i} = P_{i}^{(m \cdot m)}$.

Hence we have

**Theorem 3.** In a $C$-reducible Finsler space the $hv$-scalar curvature $P^{(1)}$ (or $P^{(2)}$) vanishes if and only if the vector $P_{i} (= P_{i}^{(m \cdot m)})$ is orthogonal to the torsion vector $C_{i} (= C_{i}^{(m \cdot m)}$).

If we put $A_{i} = LC_{i}$ and $A^{2} = A_{i}A^{i}$ ($A^{i} = g^{ir}A_{r}$), it is easily seen that

$\mu = C_{i}C_{i} \cdot o (n + 1) = (1 / 2)L^{-1}A^{2} \cdot o / (n + 1)$.

Consequently we have
COROLLARY. In a C-reducible Finsler space with constant $A^2(=A_iA^i)$, the $h\nu$-scalar curvatures $P^{(1)}$ and $P^{(2)}$ vanish.

REMARK 8. This suggests us that the $h\nu$-curvature tensor $P_{h\nu,ijk}$ will play an important role in the investigation of Finsler spaces with constant $A^2$. Pay attention to the well-known equation $A_i=L_i[\log \sqrt{g}]$ and Deicke's theorem [4].

From (3.4) and (3.5) we obtain

$$P_{hk}^{(1)} - P_{hk}^{(2)} = (n+1)N_{hk} = A_{(hk)}[C_{k|h} + C_hG_k],$$

(3.7)  
$$N_{io}^{(1)} = N_{io}^{(2)} = 0, \quad N_{oi}^{(1)} = N_{oi}^{(2)} = 0,$$

$$N_{io} = -N_{oi} = 0.$$

THEOREM 4. If the $h\nu$-Ricci tensor $P_{ij}^{(1)}$ (or $P_{ij}^{(2)}$) of a C-reducible Finsler space is symmetric, then the Finsler space is a Berwald space.

Proof. If the tensor $P_{ij}^{(1)}$ (or $P_{ij}^{(2)}$) is symmetric, from (3.7) we have $N_{ij}=0$ and $N_{io}=0$. Consequently from (3.7) $G_i = C_{i\gamma}/(n+1)=0$. Hence from Lemma 2 it is concluded that the Finsler space is a Berwald space.

PROPOSITION 5. If the symmetric part $P_{(hh)}^{(1)}$ (or $P_{(hh)}^{(2)}$) of the $h\nu$-Ricci tensor is written in the form

(3.8)  
$$P_{(hh)}^{(1)} \text{ (or } P_{(hh)}^{(2)} \text{)} = \lambda_1 h_{hh} + \lambda_2 g_{hh}$$
as a linear combination of the angular metric tensor $h_{hh}$ and the fundamental tensor $g_{hh}$ with the scalar coefficients $\lambda_1, \lambda_2$, then $C_{i\gamma}^{(2)}=0$ holds necessarily.

Proof. Suppose that $P_{(hh)}^{(1)} = \lambda_1 h_{hh} + \lambda_2 g_{hh}$. The contraction of this by $y^k$ yields

$$P_{(ko)}^{(1)} = \lambda_2 y_k.$$

Contracting the above by $y^k$, we have $\lambda_2=0$ and $P_{(ko)}^{(1)}=0$. Thus (1) of Proposition 3 gives $C_{i\gamma}^{(2)}=0$.

Consequently from Lemma 2 quoted above and Proposition 5, it is concluded that

THEOREM 5. If, in a C-reducible Finsler space, the symmetric part $P_{(hh)}^{(1)}$ (or $P_{(hh)}^{(2)}$) of the $h\nu$-Ricci tensor is written as (3.8), then the Finsler space is a Berwald space.

COROLLARY. A C-reducible Finsler space is a Berwald space if one of the following conditions holds good:

3) See [14]; [12] Th. 30.4: If a C-reducible Finsler space is a Landsberg space, then it is a Berwald space.
(1) $P_{\dot{h}h}^{(1)}$ (or $P_{\dot{h}h}^{(2)}$) is proportional to $h_{hk}$.
(2) $P_{\dot{h}h}^{(1)}$ (or $P_{\dot{h}h}^{(2)}$) is proportional to $g_{hk}$.
(3) $P_{\dot{h}h}^{(1)}$ (or $P_{\dot{h}h}^{(2)}$) vanishes.

Remark 9. In a 2-dimensional Finsler space the $h_v$-torsion tensor $C_{ijk}$ is always written as (3.1) and

$$P_{ij}^{(1)} = P_{ij}^{(2)} = P_{ij}, \quad P^{(1)} = P^{(2)} = 0$$


Theorem 6. If the $h_v$-Ricci tensor $P_{ij}$ is symmetric in a 2-dimensional Finsler space $F^2$, then $F^2$ is a Landsberg space.

Proof. The $h_v$-curvature tensor $P_{hijk}$ of $F^2$ is always written in the form

$$P_{hijk} = \sigma (l_i m_j - l_j m_i) m_j m_k,$$

where $(l_i, m_i)$ is the Berwald frame ([12] §28). The two unit vectors $l_i$ and $m_i$ are orthogonal to each other. Contracting the above equation by $g^{ij}$ we have $P_{hj}^{(1)} = \sigma l_h m_j$. The $h_v$-Ricci symmetry yields immediately $\sigma = 0$ and we get $P_{hijk} = 0$, which is equivalent to $P_{ij} = 0$. Then the proof is complete.

Next we shall consider the Bianchi identities of $C$-reducible Finsler space.

From (3.1) and $S_{i j k} [R_{ij k}] = 0$ the equation (1.9) is rewritten as

$$R_{hk} - R_{hk} = [(n-2) R_{rkh} - C_{h} R_{ok} + L^{-1} l_h R_{rko}] / (n+1).$$

The contraction of (1.9C) by $\gamma_h$ yields

$$R_{ko} - R_{ok} = [n-1] R_{rko} + C_{h} R_{ko} / (n+1).$$

From (2.1C) we get the form of $R_{rko} (= - R_{ko})$ in terms of the $h$-Ricci tensor. Substituting this into (1.9C) we obtain

$$(1.9C)' R_{hk} - R_{hk} = \frac{1}{(n-1)} A_{h} (l_h (R_{ko} - R_{oh}))$$

$$= \frac{1}{n+1} [(n-2) R_{rkh} + C_{h} T_h - C_{h} T_k],$$

where we put

$$T_h = R_{oh} - L^{-1} R_{oo} l_h / (n-1).$$

Next we reconsider (2.2)'. Using (1) of Prop. 3 and (3.3), it is rewritten in the form

$$(2.2C)' R_{ok} l_i - R_{oi} l_k = R_{r} l_{i} r + (n-2) G_r R^{r}_{ki} - A_{h} (L^{-1} G_r R_{r} l_i + R_{ok} G_l).$$

4) See [12] §25, [6].
The contraction of (2.2C)' by \( \gamma^k \) yields

\[
(2.2C)'' = R_{oo}i + R_{oi}o = R'_{oo}i + (n-1) R'_{oi}o + R_{oo}G_i = R_{oo}G_i.
\]

Next we are concerned with (2.5)'. First from (3.3) we derive

\[
P_{ki} = G^r_{ki} h_{ri} + G_{ri} h_{ki} + G_{kri} - L^{-1}(G_{ki} o^y + G_{yki} o_i).
\]

By virtue of the above, (3.1) and (3.3), the equation (2.5)’ is rewritten, after long computation, in the form

\[
(2.5C)' = R_{ok} + C_{i o k} = S_{ik} [G_{i} h_{k} - L^{-1} G_{i o} o_k + (n-3) G_{k} G_i / 2
\]

\[
- (R_{ok} - L^{-1} R_{oo} o_k) C_i / (n+1)] + \nu h_{ik} + U_{ik},
\]

where \( S_{ik} \) means the interchange of indices \( i, k \) and summation and we put

\[
\nu = G^r_{i r} + (n-1) \zeta - R_{oi} / (n+1), \quad \zeta = G r G_i,
\]

\[
(3.9)
\]

It is observed that the right hand side of (2.5C)' is symmetric except the term \( U_{ik} \).

The contraction of (2.5C)' by \( \gamma^k \) yields

\[
(2.5C)'' = R_{oo}i + C_{i o o} = R_{oi} + R_{oo} - (2 R_{oi} + R_{oo} C_i) / (n+1).
\]

From (2.5C)' we have

**PROPOSITION 6.** In a C-reducible Finsler space, if the tensor \( U_{ij} \) given by (3.9) is symmetric, then the tensor

\[
R_{oi} + C_{j o o}
\]

is symmetric.

**REMARK 10.** In a Riemannian space the term \( U_{ik} \) is, of course, equal to \( R_{ik} \) which is symmetric. Then \( R_{oi} + C_{j o o} \) is nothing but \( R_{ji} \).

**REMARK 11.** In an \( h \)-isotropic Finsler space \( F^* \) (See § 4), the above \( U_{ij} \) is written in the form

\[
U_{ij} = R_{ij} - R_{ji} C_i
\]

and \( R_{ij} \) is symmetric. Hence, if \( U_{ij} \) is symmetric, we get \( C_i = 0 \) provided \( R \neq 0 \). Consequently \( F^* \) is a Riemannian space from (3.1). In this case \( U_{ij} \) is nothing but the Ricci tensor \( R_{ij} \) in the Riemannian space.

**REMARK 12.** In a Finsler space of scalar curvature \( K \) (See § 5), it is observed that \( R_{ij} = R_{ji} \) (See § 5 Th. 9(3)). And \( U_{ij} \) is written in the form

\[
U_{ij} = R_{ij} + \frac{L^2}{3(n+1)} (K_{il} h_{ij} + K_{il} C_j + K_{il} C_i) - C_i (L^2 K_{lj} / 3 + Ky_j).
\]
If $U_{ij}$ is symmetric, we have $K=0$. Conversely, in a space of vanishing scalar curvature $K$, $U_{ij}(=R_{ij})$ is always symmetric.

Contracting $U_{ik}$ in (3.9) by $y^j$ and $y^k$, we get respectively

\begin{equation}
U_{ok}=R_{ok}, \quad U_{ko}=R_{ko}-(2R_{rok}+C_kR_{oo})/(n+1).
\end{equation}

Next we shall consider another condition $U_{ok}=U_{ko}=0$. From (3.10) we have

\begin{equation}
R_{oi}=0, \quad R_{io}=2R_{roi}/(n+1).
\end{equation}

Here we shall consider the equation (2.6)”, which is, from (3.1), written as

\begin{equation}
C_{ij0}=R_{fjo}-R_{ofj}+(2R_{roi}-R_{mo}C_{ij})/(n+1).
\end{equation}

From (3.11) the equation (2.5C)” reduces to $C_{ij0}=0$ and (2.6C)” also reduces to

\begin{equation}
R_{fjo}=R_{fjo}.
\end{equation}

Furthermore we are concerned with more stronger condition $U_{ij}=0$. In this case (2.5C)” is written as follows:

\begin{equation}
C_{ij0}=G_{i1k}+G_{kj1}+(n-3)G_{i}G_{k}+\kappa h_{ik}, \quad (\kappa=G_{11}+(n-1)\zeta).
\end{equation}

Hence, by virtue of $C_{ij0}=(n+1)G_{ij}$, the above, which also means $G_{i1k}=G_{kj1}$, is rewritten in the form

\begin{equation}
G_{k1i}=(n-3)G_{k}G_{i}+\xi h_{ki} / (n-1).
\end{equation}

Contracting this by $g^{ki}$ we get $\zeta=0$. Consequently the above reduces to a little simple form

\begin{equation}
G_{k1i}=(n-3)G_{k}G_{i}+\xi h_{ki} / (n-1), \quad (\xi=G_{1i}).
\end{equation}

If the metric is positive-definite, $\zeta=0$ implies $G_{i}=0$.

Thus, making a summary of the results obtained above, we have

**Theorem 7.** In a C-reducible Finsler space with $U_{ok}=U_{ko}=0$ the following hold good:

1. The contracted Ricci tensors $R_{oi}$ and $R_{io}$ satisfy the equation (3.11),
2. $C_{ij0}=0$,
3. the contracted $(v)h$-torsion $R^r_{fjo}$ satisfies the equation (3.12).

**Corollary.** In a C-reducible Finsler space with $U_{ij}=0$, the conditions (1), (2) and (3) of Theorem 7 hold good and further

4. the tensor $G_{k1i}$ is symmetric and written as (3.13),
5. the space is a Berwald space, provided the metric be positive-definite.
Here we shall recollect the Proposition 4. In virtue of \( h_i^j |_i = -(n-1) L^{-1} l_k \) and \( G_i |_o = 0 \), it follows from (3.3) that
\[
P_i r_j |_r = G_i |_r h_r^j - S_{i(jp)} \{ n L^{-1} G_j |_j - G_i |_i \}.
\]
On the other hand from (3.1) we have
\[
C_i r_j |_r = C_i r h_r^j / (n+1) + \frac{1}{(n+1)} S_{i(jp)} \{ C_i |_i j - (n+1) L^{-1} l_p G_j \}.
\]
Consequently (2.9) is of the form
\[
C_i r_j |_r - C_i |_r j = \omega h_r^j - S_{i(jp)} \{ (n-1) L^{-1} l_p G_j - G_i |_i j + C_i |_i j / (n+1) \},
\]
which is rewritten as
\[
(2.9C) \quad C_i r_j |_r - C_i |_r j = (n+1) \{ \omega h_r^j - L^{-1} (l_i G_j + l_j G_i) \},
\]
\[
(\omega = G_i |_r - C_i r / (n+1)).
\]
Hence we have

**PROPOSITION 4C.** In a C-reducible Finsler space the symmetric tensor \( C_i r_j |_r - C_i |_r j \) - \( C_i |_i j \) is given by (2.9C).

§ 4. Isotropic Finsler spaces.

We shall consider an \( h \)-isotropic Finsler space \( F^* \) which is introduced by H. Akbar-Zadeh ([1], [12] § 22).

In the isotropic Finsler space the \( h \)-curvature tensor \( R_{hi}jk \) is written in the form
\[
(4.1) \quad R_{hi}jk = R(g_{hk}g_{ij} - g_h g_{ij}).
\]

**THEOREM (H. Akbar-Zadeh)** An \( h \)-isotropic Finsler space of dimension \( n \geq 3 \) are such that

(1) \( R = \text{constant} \).
(2) \( P \)-symmetry and \( S_{hi}jk = 0 \), provided \( R \neq 0 \).

From (4.1) it is easily verified that the Ricci tensor \( R_{ij} \) is symmetric, i.e., \( R_{ij} = (n-1) R g_{ij} \). The equations (2.2), (2.2)', (2.2)" are trivial because of \( R = \text{constant} \) and metrical property \( g_{ij} |_k = 0 \) of the Cartan connection \( CF \).

Here we shall reconsider (2.5)" and (2.6)' which were derived from the Bianchi identities of the second group. From (4.1) we have
\[
R_{ij} = R (y_j g_{ik} - y_k g_{ij}).
\]
Consequently (2.5)" reduces to
Next (2.6)' is now rewritten as

$$C_{j|o|k} - C_{k|o|j} = R(y_j C_k - y_k C_j).$$

REMARK 13. The contraction of (2.6)' by $y^k$ yields (2.5l)' immediately. From (2.5l)'' we see that $C_{i|o|o} = 0$ is equivalent to $C_i = 0$, provided $R \neq 0$. This fact has been shown already by H. Akbar-Zadeh ([1] p. 48).

REMARK 14. An $h$-isotropic and $C$-reducible Finsler space is a Riemannian space (See [14] Theorem 2).

Hereafter suppose that $R \neq 0$. Then we see $P_{ij}^{(1)} = P_{ij}^{(2)}$ because of the $P$-symmetry. We shall denote it by $P_{ij}$. The expressions (2.6) and (2.10) are written respectively as follows:

(2.6)

$$P_{ij|k} - P_{ik|j} = A(jk) \{ R g_{ij} C_k + P_{t' jr} P_{t' k} \},$$

(2.10)

$$P_{ij|k} - P_{k|ij} = P_{t' k} C'_{j} - P_{t' r j} C'_{i} R.$$  

From these equations we get respectively

$$P^m_{j|l|m} = - (n-1) R C_{i|j|m} P^m_{l|P_{r|m}},$$

$$P^m_{i|l|m} = - P^m_{r} C'_{i|l|m}. $$

§ 5. The scalar curvature and Ricci tensors.

A Finsler space $F^n$ is said to be of \textit{scalar curvature} $K$ ([2], [12] § 26) if the equation

$$R_{i|o|j} = KL^2 h_{ij}$$

holds good at any $(x, y)$ of $F^n$, and to be of \textit{constant curvature} $K$ if, furthermore, the scalar $K$ is constant.

It is well-known ([12] (26.5)) that (5.1) is equivalent to

$$R_{ijk} = L^2 (K h_{ik} - K h_{ij}) / 3 + K (y_j h_{ik} - y_k h_{ij}),$$

where we denote $\frac{\partial}{\partial y^i}$.

In general the Berwald curvature tensor $H_{hijk}$ satisfies the following identities ([12] § 18):

$$H_{hijk} = R_{ijk|h} - 2 C_{irk} R'_{jk},$$

$$H_{hijk} = - H_{hikj},$$

$$S_{ijk} [H'_{h} kj] = 0.$$ 

Substituting (5.2) into (5.3), we have ([12] (26.6))
Here we shall define three $H$-Ricci tensors from the Berwald curvature tensor $H_{i\tilde{j}k}$ in the following form:

\begin{equation}
H_{ij}=H_{i}^{\prime}j_{r}, \quad H_{ij}^{(1)}=H_{ir}j_{r}, \quad H_{ij}^{(2)}=H_{r}j_{i}.
\end{equation}

From (5.5) it is easily seen that $H_{ij}-H_{ji}=H_{ji}^{(2)}$.

Next it follows from (5.6) that

\begin{equation}
H_{ij}=(n-1)(K_{ij}+K_{i\tilde{j}}+K_{\tilde{i}j})+(n-2)L^{2}K_{i\tilde{i}j}j
\end{equation}

\begin{equation}
+(n+1)K_{i\tilde{j}}y_{j}/3,
\end{equation}

\begin{equation}
H_{ij}^{(1)}=(n-1)K_{ij}+(n-3)(K_{i\tilde{j}}j_{i}+K_{\tilde{i}j}y_{j})+L^{2}K_{i\tilde{j}m}g_{mn}h_{ij}
\end{equation}

\begin{equation}
-(n+1)K_{i\tilde{j}}y_{j}/3.
\end{equation}

From (5.8), (5.9) the $H$-scalar curvatures $H (=H_{ij}g^{ij})$ and $H^{(1)} (=H_{ij}^{(1)} g^{ij})$ are written in the form

\begin{equation}
H=n(n-1)K+(n-2)L^{2}K_{i\tilde{i}j}g_{ij}/3,
\end{equation}

\begin{equation}
H^{(1)}=H.
\end{equation}

On the other hand, from (5.8), (5.9) it is observed that

\begin{equation}
H_{ij}-H_{ji}=H_{ji}^{(1)}-H_{ij}^{(1)}=H_{ji}^{(2)}=(n+1)(K_{i\tilde{j}}j_{i}-K_{\tilde{j}i}j_{i})/3.
\end{equation}

**Theorem 8.** Let $F^{n}$ $(n\geq3)$ be a Finsler space of scalar curvature $K$. Then $F^{n}$ is of constant curvature if and only if the Ricci tensor $H_{ij}$ (or $H_{ij}^{(1)}$) is symmetric or $H_{ij}^{(2)}$ vanishes.

**Proof.** From (5.11) we have $K_{i\tilde{j}}j_{i}-K_{\tilde{i}j}j_{i}=0$. Contracting this by $y_{j}$, we get $K_{i\tilde{i}}=0$, which means that $K$ is a function of position only. By virtue of generalized Schur's theorem, $K$ becomes constant (See [2], [12] Prop. 26.1). The converse is clear.

**Remark 15.** This theorem as to $H_{ij}$ has been shown by L. Berwald ([2] p. 775).

The relation between the Cartan's $h$-curvature tensor $R_{hi\tilde{j}k}$ and the Berwald's one $H_{hi\tilde{j}k}$ is generally given by

\begin{equation}
R_{hi\tilde{j}k}=(H_{hi\tilde{j}k}-H_{hi\tilde{j}k})/2-P_{hr}P_{j}k+P_{kr}P_{j}r.
\end{equation}

Next we shall treat the $h$-Ricci tensor $R_{ij}$ in the space of scalar curvature. Substituting (5.6) into (5.12) we get

\begin{equation}
R_{hi\tilde{j}k}=K(h_{h\tilde{j}}h_{ik}-h_{h\tilde{j}}h_{ij})+A_{ijk}(h_{ik}M_{kj}-h_{h\tilde{j}}M_{ij})
\end{equation}
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where we put

\[ M_{hj} = M_{jh} = (L^2 K_{h}^{j} + 3 K_{h}^{j} + 3 K_{h}^{j} + 6 K_{h}^{j}) / 6. \]

Multiplying (5.13) with \( g^{hk} \) we have

\[ R_{ij} = (n-1) 6 K_{g_{ij}} + (3n-7) (K_{g_{ij}} + K_{jg_{i}} - 3 K_{g_{ij}} - 3 K_{jg_{i}} + 6 K_{g_{ij}}) / 6 + P_{m}^{i} P_{m}^{n} - P_{m}^{i} P_{m}^{n}. \]

From (5.14) the h-scalar curvature \( R (= R_{ij} g^{ij}) \) is given by

\[ R = \{3n(n-1) K_{g_{ij}} + (n-2) L^2 K_{g_{ij}} + n K_{g_{ij}} \} / 3 + g^{ij} P_{m}^{i} P_{m}^{n} - P_{m}^{i} P_{m}^{n}. \]

We have from (5.13) and (5.14) directly

**Theorem 9.** In a Finsler space of scalar curvature \( K \) the following hold good:

1. \( R_{hij} = R_{jhi} \),
2. \( S_{i}^{i} [R_{hjk}] = 0 \),
3. The h-Ricci tensor \( R_{ij} \) is symmetric and given by (5.14).

**Remark 16.** In a Finsler space of scalar curvature it is verified by means of (5.2) that

\[ S_{i}^{i} [C_{r}^{i} R_{rjk}] = 0, \]

\[ A_{i}^{i} [C_{r}^{i} R_{rhk} - C_{r}^{i} R_{rkh}] = 0. \]

Therefore from (1.7), (1.8) we have another proof of (1), (2) of Theorem 9 not referring to the components of the h-curvature tensor \( R_{hijk} \) of (5.13).

**Remark 17.** In case of constant curvature, the Ricci tensor \( R_{ij} \) is, of course, symmetric as shown and used by Y. Takano [25]. In case of scalar curvature, (3) of Theorem 9 is shown independently and almost simultaneously by C. Shibata [21].

**Theorem 10.** Let \( F^n(n \geq 3) \) be a Finsler space of scalar curvature \( K \). If \( R_{ij} = \gamma g_{ij} \) with some scalar \( \gamma \), then \( F^n \) is of constant curvature.

**Proof.** Suppose that \( R_{ij} = \gamma g_{ij} \). Contracting this by \( y^i \), we get \( R_{ii} = \gamma y_i \). From (5.14) we have

\[ R_{oi} = \{3(n-1) K y_i + (n-2) L^2 K_{i} y_i \} / 3. \]

These equations yield

\[ 3(n-1) K y_i + (n-2) L^2 K_{i} y_i = 3 \gamma y_i. \]
We take the contraction of the above by $\gamma^i$ to get $\nu = (n-1)K$, so that $K_{ij} = 0$ provided $n \geq 3$. Consequently $K$ is constant.

**Theorem 11.** Let $F^n (n \geq 3)$ be a Finsler space of scalar curvature $K$. If $H_{ij} = \nu g_{ij}$ (or $H_{ij}^{(\nu)} = \nu' g_{ij}$) with some scalar $\nu$ (or $\nu'$), then $F^n$ is of constant curvature, where $(\ )$ denotes the symmetric part of $H_{ij}$ (or $H_{ij}^{(\nu)}$).

This is easily obtained in the similar way to the proof of Theorem 10.

Here we recall Theorem 1 in §2. In case of scalar curvature, it is easily seen that the quantity $Z_i$ (See (2.12)) is identically zero. Further, from (5.2), (5.14) the tensor $B_i^h$ is written in the form

$$B_i^h = (n-2)L^2 Z_i^h/3, \quad (Z_i^h = K_{ij} y^j - 3K h_i^j, \quad h_i = h_{ij} g^{kj}).$$

Hence we have from Theorem 1

**Theorem 12.** In an $n$ ($\geq 3$)-dimensional Finsler space of scalar curvature $K$, the tensor $Z_i^h = K_{ij} y^j - 3K h_i^j$ satisfies the conservation law $Z_i^h = 0$.

**Remark 18.** In case of constant curvature $K$, from (EC-1-3) H. Rund has derived ([18] (3.15), [12] Th. 26.4) another conservational law $G_i^h = 0$, where

$$G_i^h = R_i^h - R_0^h = 2 - K S y_i^j/2, \quad (S = S_{ij} g^{ij}).$$

**Remark 19.** In a Finsler space of scalar curvature $K$ the equation, which is one of the Bianchi identities of the Berwald connection $B\Gamma$ ([12] (18.20)),

$$S_{ijkl} \{G_m^{ik} R_{jk} + H_m^{kj} h_{ij} \} = 0$$

reduces to a simple form

$$S_{ijkl} \{H_m^{kj} h_{ij} \} = 0.$$

This was suggested by Prof. M. Matsumoto. The semicolon (;) means the covariant differentiation with respect to the Berwald connection $B\Gamma$. The contraction of the above equation by $g^{mk}$, however, leads us rather complicated equation, because the Berwald connection $B\Gamma$ is not metrical.

Next we are concerned with a $C$-reducible Finsler space of scalar curvature $K$. In a $C$-reducible Finsler space, $P_{ijk}$ is given by (3.3). Substituting this into (5.13), we obtain

$$R_{ijk} = A_{ijk} \{h_{ik} N_{jk} + h_{kj} N_{ki} \},$$

where we put $N_{ij} = (K - \zeta) h_{ij}/2 + M_{ij} - G_i G_j$. The expression (5.16) is very interesting because the $h$-curvature tensor $R_{ijk}$ is simply written in terms of an angular metric tensor $h_{ij}$. Cf. Theorem 29.2 of [12]. Similarly substituting (3.3) into (5.14), (5.15), we get respectively
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(5.17) \[ R_{ij} = (n-1)(K - \zeta) + L^2 K_{ilm} g^{mn}/6 \cdot h_{ij} + (n-1)K I_{ij} \]
\[ + (3n-7) L(I_{ijkl} + I_{jkl}) + (n-3) L^2 K_{ijkl} /6 - (n-3)G_iG_j, \]

(5.18) \[ R = R_{ij}g^{ij} = n(n-1)K - (n-2) (n+1) \zeta - L^2 K_{ilm} g^{mn}/3. \]

Consequently, in case of \( K=0 \), we have

**Theorem** (H. Yasuda [25] Th. 7). Let \( F^n (n \geq 3) \) be a C-reducible Finsler space of vanishing scalar curvature \( K \). Then \( F^n \) becomes locally Minkowski\(^5\) if one of the following conditions holds good:

1. In case of \( n=3 \), \( R_{ij} = 0 \) and positive-definite.
2. In case of \( n \geq 3 \), \( R=0 \) and positive-definite.
3. In case of \( n>3 \), \( R_{ij} = 0 \).

**Proof.** Suppose that \( K=0 \). Then (5.17) and (5.18) reduces to respectively

(5.19) \[ R_{ij} = -(n-1) \zeta h_{ij} - (n-3)G_iG_j, \]
(5.20) \[ R = -(n-2)(n+1) \zeta. \]

If \( R_{ij} = 0 \) and \( n=3 \), from (5.19) we obtain \( \zeta (=G_iG_j) = 0 \), so that \( G_i = 0 \) (the Berwald space), provided the metric is positive-definite. Consequently we have \( N_{hj} = 0 \), so that, from (5.16), \( R_{hijk} = 0 \). If \( R_{ij} = 0 \) and \( n>3 \), we contract (5.19) for \( i \) and \( j \) to obtain \( \zeta = 0 \), so that we have \( G_i = 0 \). The proof of (2) is similarly obtained from (5.20).

**Remark 20.** The \((\nu) h\nu\)-torsion tensor \( P_{ijk} \) takes place in (5.13), (5.14) and (5.15). We have treated these equations for C-reducible spaces where \( P_{ijk} \) is of a simple form (3.3). The simplest case \( P_{ijk} = 0 \) leads us to a trivial result, because S. Numata ([17], [12] §30) has shown that a Finsler space \( (n \geq 3) \) of scalar curvature \( K \neq 0 \) with \( P_{ijk} = 0 \) is a Riemannian space of constant curvature \( K \).

**§ 6. The \( v \)-curvature tensor \( S_{hijk} \) and the Ricci tensor \( S_{ij}. \)**

We consider a tangent space \( F^*_x \) of an \( n \)-dimensional Finsler space \( F^n \) at a point \( x=(x^i) \). Then \( F^*_x \) is regarded as an \( n \)-dimensional Riemannian space equipped with the fundamental tensor \( g_{ij} (x, y) \) where \( x \) is fixed. The components of the C-tensor \( C^i_{jk} \) are nothing but the Christoffel symbols constructed from \( g_{ij} (x, y) \) with respect to \( y \) and the \( v \)-curvature tensor \( S_{hijk} \) is the Riemannian curvature tensor of \( F^*_x \). Consequently the tensor \( U_{i}^j = S^l_{i} - \delta^l_i S_{ij}/2 (S^l_{ij} = g_{jr}S_{ri}, \ S_{ij} = g_{mn}S_{imjn}, \ S = g_{ij}S_{ij}) \) satisfies the conservation law \( U_{i}^j |_j = 0 \) which is the well-known result in Riemannian geometry [22].

The \( v \)-curvature tensor \( S_{hijk} \) and the \( v \)-Ricci tensor \( S_{ij} \) are studied by, for

\(^5\) If \( R_{hijk} = 0 \) and \( C_{ijk} = 0 \), then the Finsler space is called *locally Minkowski* ([12] §24).
instance, F. Brickell [3], S. Kikuchi [5], M. Matsumoto [9], [11], [12],
S. Numata [16] and others [15], [23], [24].

A Finsler space $F^n$ is called $(\alpha, \beta)$-metric if the fundamental function is
of the form $L(\alpha, \beta)$, where $\alpha^2(x, y) = a_{ij}(x)y^i y^j$, $\beta(x, y) = b_i(x)y^i$ and $L(\alpha, \beta)$ is positively homogeneous of degree 1 in $\alpha, \beta$. Here the quadratic form $\alpha^2(x, dx)$ is supposed to be a Riemannian metric of the space. If $L(\alpha, \beta)$ is of the form 
$L(\alpha, \beta) = \alpha + \beta$ (resp. $\alpha^2/\beta$), the Finsler space is called the Randers space (resp. Kropina space). The concrete form of the $v$-curvature tensor $S_{hijk}$ of the Randers space is seen in [8], [19]. The one of the Kropina space is given by C. Shibata [20]. Further in case of $(\alpha, \beta)$-metric, which is called the generalized Randers space, S. Numata gives the $v$-curvature tensor $S_{hijk}$ in a very simple form [16] and obtains the $v$-Ricci tensor $S_{ij}$. Here we sum up the results obtained hitherto related to the $v$-curvature tensor $S_{hijk}$ and the $v$-Ricci tensor $S_{ij}$.

(I) In case of $n=2$

The $v$-curvature tensor $S_{hijk}$ is identically zero ([10] p. 152, [12] Prop. 28 3).

(II) In case of $n=3$

(1) The $v$-curvature tensor $S_{hijk}$ is always written in the following form [5], [11], [12]
\[
L^2 S_{hijk} \parallel S(h_{ik}h_{kj} - h_{ki}h_{kj})
\]
where $S$ is some $(0)p$-homogeneous scalar field in $y^i$. Consequently the $v$-Ricci tensor $S_{ij}$ is given by
\[
S_{ij} = L^{-2} S_{hij}.
\]

(2) The $v$-curvature tensor $S_{hijk}$ vanishes if and only if the $v$-Ricci tensor $S_{ij}$ vanishes. Consequently the Finsler space $F^3$ with $S_{ij}=0$ is a Riemannian space under the well-known F. Brickell's conditions [3].

(3) If $R_{ij} = \nu g_{ij}$ with some scalar $\nu$, then the $v$-curvature tensor $S_{hijk}$ or the $h$-curvature tensor $R_{hijk}$ vanishes [9].

(III) In case of $n \geq 3$

(1) The $v$-curvature tensor $S_{hijk}$ vanishes if and only if the indicatrix $I_x$ is of constant curvature 1 ([12] Th. 31.1).

(IV) In case of $n=4$

(1) The $v$-curvature tensor $S_{hijk}$ is always written in the form
\[
S_{hijk} = A_{(ijk)} \{ h_{kj}M_{jk} + h_{ik}M_{kj} \},
\]
where $M_{ij} = S_{ij} - Sh_{ij}/4$, $S = S_{ij}g^{ij}$ ([11], [12] Th 31.2).

(V) **In case of $n \geq 4$**

1. Let $F^n$ be a Finsler space with $(\alpha, \beta)$-metric. If $S_{ij} = 0$, then $F^n$ is a Riemannian space provided $\beta^2 \neq \text{constant}$ [16].

2. Let the $v$-curvature tensor $S_{hijk}$ is of the form (6.1). Then the scalar $S = S(x)$ and the indicatrix $I_x$ is of constant curvature $S+1$ ([12] Th. 31.6).

3. Suppose that the $v$-curvature tensor $S_{hijk}$ is of the form (6.4)

   $S_{hijk} = A_{ijk} \{ h_{kj}E_{ik} + h_{ik}E_{kj} \}$

   where $E_{ij}$ is a $(-2)p$-homogenous Finsler tensor field. Then the followings hold good [15]:

   (1°) $E_{ij}$ is given by

   $E_{ij} = \frac{1}{n-3} \left\{ \frac{S}{2(n-2)} h_{ij} - S_{ij} \right\}$

   and (6.4) is written in the form

   (6.4)'

   $S_{hijk} = \frac{S}{(n-2)(n-3)} \left( h_{kn}h_{ij} - h_{kj}h_{ik} \right) - \frac{1}{(n-3)} A_{ijk} \{ h_{kj}S_{ij} - h_{ik}S_{kj} \}$

   (2°) If $S_{ij} = 0$, then $S_{hijk} = 0$.

   (3°) $F^n$ is $P$-symmetry if and only if $S_{ij} = 0$.

(VI) **In case of $n \geq 5$**

1. Let the $v$-curvature tensor $S_{hijk}$ is of the form (6.4). Then the following hold good:

   (1°) The indicatrix $I_x$ is conformally flat, i.e., the Weyl conformal curvature tensor vanishes [12].

   (2°) The tensor

   $h_{ij}T_k + (n-2)(S_{ik}T_j + L^{-1}S_{ik}T_j)$

   is symmetric in $j$ and $k$, where $T_k = S_{ik}/2 + L^{-1}S_{ik}$ [15].

**Remark 21.** In the paper [9], quasi-C-reducible Finsler spaces with $S_{ij} = 0$ are also treated. Any Finsler spaces with $(\alpha, \beta)$-metric is quasi-C-reducible. With respect to the contracted tensor $T_{ij} = g^{mn}T_{imjn}$, S. Numata proved the following:

1. Let $F^n (n \geq 3)$ be a Finsler space with $(\alpha, \beta)$-metric. If $T_{ij} = 0$, then $F^n$ is a Riemannian space provided $\beta^2 \neq \text{constant}$, where the original tensor $T_{hijk}$ is defined in the form
Recently this tensor $T_{ijkl}$ is used by Prof. M. Matsumoto in characterizing Finsler spaces with cubic metric.

References

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