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A NOTE ON IDEAL STRUCTURE AND UNIT GENERATION IN SOME FULLY IDEMPOTENT AND REGULAR RINGS

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This note was prompted by recent work on the ideal structure of some regular rings c. f. [2, 4] and by an older interest in unit generation which goes back to Shornjakov [5] (c. f [6], [3] etc). The simplest example of the rings we will be discussing is the ring of endomorphisms of a vector space because its ideals form a well ordered set under inclusion.

A ring is called fully idempotent [1] if each of its two-sided ideals coincides with its own square. It is equivalent to ask that each homomorphic image of the ring be semiprime. Regular rings, biregular rings, and V-rings are examples of fully idempotent rings.

PROPOSITION 1. *Let R be fully idempotent ring then the following are equivalent:*

(i) *the ideals of R are totally ordered under inclusion,*

(ii) *the prime ideals of R are totally ordered under inclusion*

(iii) *each ideal of R is prime.*

PROOF. Clearly (i)$\Rightarrow$(ii). Assume (ii) and let I be an ideal of R. I is semiprime so $I=\bigcap P_\alpha$ for some prime ideals $\{P_\alpha\}$. Let J, K be ideals with $J\supsetneqq I$, $K\supset I$ and $JK\subset I$. Then for some α, $J\not\subset P_\alpha$. For any P_β in $\{P_\alpha\}$, one has $P_\beta\subset P_\beta$ or $P_\beta\not\subset P_\alpha$. If $P_\beta\subset P_\alpha$, $J\not\subset P_\beta$ so $K\subset P_\beta$. If $P_\beta\not\subset P_\alpha$, $P_\alpha\subset P_\beta$ by (ii) and $K\subset P_\beta$ since $K\subset P_\alpha$. Thus $K\subset I$. (iii)$\Rightarrow$(i). Let I and J be deals of R with $I\not\subset J$. IJ is a prime ideal and $I\not\subset IJ$ so $J\subset IJ\subset I$.

PROPOSITION 2. *Let R be a regular ring in which the ideals form a well-ordered set under inclusion, for example, a prime regular right self-injective ring or a homorphic image of such a ring. Then there exists in R a set $\{e_\alpha\}$ of idempotents, which is indexed by the ideals corresponding to the non-limit ordinals indexing the ideals such that if I_β is an ideal indexed by the ordinal β,*

$I_\beta=Re_\alpha R$ for some α if β is a non-limit ordinal, and $I_\beta=\sum_{\alpha<\beta} Re_\alpha R$ if β is

a limit ordinal. Furthermore the set $\{e_\alpha\}$ is minimal with respect to having this property.

PROOF. Let x be a non-unit in R and let I be the first ideal to contain x. $xR=eR$ for some idempotent e and $x\in ReR\subset I$ so $ReR=I$. Now let I correspond to the non-limit ordinal α and relabel e, e_α. This process yields a set $\{e_\alpha\}$ with the required properties.

The following result is of intersted from the point of view of unit generation.

LEMMA 3. (Stephenson) *Let R be a ring and let e an idempotent element of R. Then every element of the form $res(1-e)t$ can be expressed in terms of sums of products of nilpotent elements.*

PROOF. $res(1-e)t = (eres(1-e)(1-e)te)+eres(1-e)t(1-e)+(1-e)res(1-e)te +((1-e)re)(es(1-e)t(1-e))$. The second and third terms are nilpotents, and the first and fourth are products of two nilpotents. This gives.

PROPOSITION 4. *Let R be a regular ring in which the ideals are totally ordered by inclusion. Then each ideal of R is generated by the nilpotents it contains.*

PROOF. Let I be an ideal in such a ring and let $x\in I$. $xR=eR$ for some e and $ReR(1-e)R+R(1-e)ReR\subset I$. Each nilpotent that appears in lemma 3 lies in one of these two ideals for x can be expressed in terms of nilpotents in I.

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