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# ON SASAKIAN MANIFOLDS WITH VANISHING $C$-BOCHNER CURVATURE TENSOR

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Recently, S.I. Goldberg [1] proved

THEOREM A. *Let $M$ be an $n$-dimensional ($n \geqq 3$) compact conformally flat Riemannian manifold with constant scalar curvature. If the length of the Ricci tensor is less than $K/\sqrt{n-1}$, then $M$ is a space of constant curvature.*

Also, S.I. Goldberg and M. Okumura [2] proved

THEOREM B. *Let $M$ be an $n$-dimensional ($n \geq 3$) compact conformally flat Riemannian manifold. If the length of the Ricci tensor is constant and less than $K/\sqrt{n-1}$, then $M$ is a space of constant curvature.*

In 1976, Y. Kubo [3] proved the following theorems corresponding to those of Golderg-Okumura, replacing the vanishing of the Weyl conformal curvature tensor of a Riemannian manifold by that of the Bochner curvature tensor of a Kaehlerian manifold.

THEOREM C. *Let $M$ be a Kaehlerian manifold of real dimension $n (n \geq 4)$ with constant scalar curvature whose Bochner curvature tensor vanishes. If the length of the Ricci tensor is not greater than $K/\sqrt{n-2}$, then $M$ is a space of constant holomorphic sectional curvature.*

THEOREM D. *Let $M$ be a Kaehlerian manifold of real dimension $n (n \geq 4)$ whose Bochner curvature tensor vanishes. If the length of the Ricci tensor is constant and not greater than $K/\sqrt{n-2}$ then $M$ is a space of constant holomorphic sectional curvature.*

The purpose of the present paper is to prove the following theorems corresponding to Theorem $C$ and $D$, replacing the vanishing of the Weyl conformal curvature tensor or Bochner curvature tensor by that of $C$-Bochner curvature tensor (See [5]) in a Sasakian manifold.

THEOREM 1. *Let $M^n$ be a Sasakian manifold of dimension $n$ with constant* *scalar curvature K whose C-Bochner curvature tensor vanishes. If the square of the length of the Ricci tensor is less than* $K^2/n-1$*, then* $M^n$ *is locally C-Fubinian.*

THEOREM 2. *Let* $M^n$ *be a compact Sasakian manifold of dimension n whose C-Bochner curvature tensor vanishes. If the square of the length of the Ricci tensor is constant and less than* $K^2/n-1$*, then* $M^n$ *is locally C-Fubinian.*

## 1. Introduction

Recently Matsumoto and Chūman [5] introduced a tensor field of type (1,3) in an $n$-dimensional Sasakian manifold $M^n$ whose components $B_{kji}{}^h$ are given by

$$
\begin{aligned}
(1.1)\quad B_{kji}{}^h = {} & K_{kji}{}^h + \frac{1}{n+3}(K_{ki}\delta_j^h - K_{ji}\delta_k^h + g_{ki}K_j^h - g_{ji}K_k^h \\
& + S_{ki}\phi_j^h - S_{ji}\phi_k^h + \phi_{ki}S_j^h - \phi_{ji}S_k^h + 2S_{kj}\phi_i^h + 2S_i^h\phi_{kj} \\
& - K_{ki}\eta_j\eta^h + K_{ji}\eta_k\eta^h - \eta_k\eta_i K_j^h + \eta_j\eta_i K_k^h) \\
& - \frac{k+n-1}{n+3}(\phi_{ki}\phi_j^h - \phi_{ji}\phi_k^h + 2\phi_{kj}\phi_i^h) \\
& - \frac{k-4}{n+3}(g_{ki}\delta_j^h - g_{ji}\delta_k^h) \\
& + \frac{k}{n+3}(g_{ki}\eta_j\eta^h + \eta_k\eta_i\delta_j^h - g_{ji}\eta_k\eta^h - \eta_j\eta_i\delta_k^h),
\end{aligned}
$$

where $\phi_j^i$ is the structure tensor, $\eta^i$ the structure vector, $g_{ji}$ the Riemannian metric tensor, $\eta_i = g_{ih}\eta^h$, $K_{kji}{}^h$ the curvature tensor, $K_{ji} = K_{hji}{}^h$ the Ricci tensor, $K = g^{ji}K_{ji}$ the scalar cuvature, $(g^{ji}) = (g_{ji})^{-1}$, and $S_{kj} = \phi_k^h K_{hj}$, $S_k{}^i = S_{ki}\, g^{ji}$ and $k = (K+n-1)/(n+1)$. They called it $C$-Bochner curvature tensor and obtained the following identities concerning with this tensor field: .

$$
\begin{aligned}
(1.2)\qquad & B_{kji}{}^h = -B_{jki}{}^h,\ B_{kjih} = B_{ihkj}, \\
& B_{kji}{}^h + B_{jik}{}^h + B_{ikj}{}^h = 0,\ B_{kji}{}^k = 0, \\
& B_{kji}{}^h\eta_h = 0,\ \phi_k^s B_{sji}{}^h = \phi_j^s B_{ski}{}^h,\ \phi^{kj}B_{kji}{}^h = 0,
\end{aligned}
$$

where $B_{kjih} = B_{kji}{}^s g_{sh}$, $\phi^{kj} = \phi_s^j g^{sk}$.

Now we shall introduce a tensor field of type (1,3) in $M^n$ whose components $U_{kji}{}^h$ are defined by

$$U_{kji}{}^h = K_{kji}{}^h - (\rho+1)(g_{ji}\delta_k^h - g_{ki}\delta_j^h)$$

$$-\rho(g_{ki}\eta_j\eta^h+\eta_k\eta_i\delta_j^h-g_{ji}\eta_k\eta^h-\eta_j\eta_i\delta_k^h+\phi_{ji}\phi_k^h-\phi_{ki}\phi_j^h-2\phi_{kj}\phi_i^h),$$

$\rho+1=\frac{k}{n-1}$, which is an analogy of the concircular curvature tensor in a Kaehlerian manifold. A Sasakian manifold $M^n$ is called *locally C-Fubinian* [11] when the tensor field $U_{kji}{}^h$ vanishes identically on $M^n$. When a Sasakian manifold $M^n$ is locally $C$-Fubinian, its Ricci tensor satisfies

$$K_{ji}=ag_{ji}+b\eta_j\eta_i,$$

where $a=\frac{K}{n-1}-1$ and $b=-\frac{K}{n-1}+n$. In this case the manifold $M^n$ is called $C$-Einstein [8]. Hence if a Sasakian manifold is locally $C$-Fubinian, then it is $C$-Einstein. Using this relation, Matsumoto and Chūman [5] proved

THEOREM E. *The C-Bochner curvature tensor $B_{kji}{}^h$ coincides with $U_{kji}{}^h$ if and only if $M^n$ is a C-Einstein space.*

By meanns of this theorem a Sasakian manifold $M^n$ with vanishing $C$-Bochner curvature sensor is locally $C$-Fubinian if $M^n$ is a $C$-Einstein space.

The present anthor [9] proved by using Theorem $E$ the following theorem:

THEOREM F. *Let $M^n$ be an n-dimensional Sasakian manifold with constant scalar curvature whose C-Bochner curvature tensor vanishes identically. If the Ricci tensor is positive semi-definite, then $M^n$ is locally C-Fubinian.*

In section 2, we shall recall fundamental properties of a Sasakian manifold with vanishing $C$-Bochner curvature tensor and in section 3 prove that the Laplacian

$$\frac{1}{2}\Delta(Z_{ji}Z^{ji})=g^{ji}(\nabla_j\nabla_i Z_{st})Z^{st}+(\nabla_k Z_{ji})(\nabla^k Z^{ji})$$

of the tensor $Z_{ji}$ defined by

$$Z_{ji}=K_{ji}-\left(\frac{K}{n-1}-1\right)g_{ji}+\left(\frac{K}{n-1}-n\right)\eta_j\eta_i, \tag{1.3}$$

$Z^{ji}=Z_{st}g^{sj}g^{ti}$, $\nabla_i$ being the operator of covariant differentiation with respect to the Riemannian connection of $M^n$ and $\nabla^k Z^{ji}=g^{ks}\nabla_s Z^{ji}$, is zero in a Sasakian manifold with constant scalar curvature whose $C$-Bochner curvature tensor vanishes identically.

In the last section 4 we shall prove the main theorems stated as before by using

Theorem A and Lemma 2.

## 2. Some properties of a Sasakian manifold with vanishing $C$-Bochner curvature tensor

Let $M^n$ be an $n$-dimensional Sasakian manifold ($n \geq 3$). Then we can easily verify that the following relations hold on $M^n$:

$$(2.1)\quad \begin{cases} S_{ji}=-S_{ij},\ \nabla_k S_j^{\ k}=\frac{1}{2}\phi_j^{\ k}\nabla_k K+(K-n+1)\eta_j\ , \\ \nabla_k S_{ji}=\eta_j K_{ik}-(n-1)g_{jk}\eta_i+\phi_j^{\ t}\nabla_k K_{ti}, \\ \phi_j^{\ t}\nabla_t S_{ik}=-\eta_i S_{kj}+(n-1)\phi_{ij}\eta_k+\phi_j^{\ r}\phi_i^{\ s}\nabla_r K_{sk} \end{cases}$$

with the help of $K_{ji}\eta^j=(n-1)\eta^j$ (See [8]). On the other hand the differential form $S=\frac{1}{2}S_{ji}dx^j\wedge dx^i$ is closed, that is,

$$\nabla_k S_{ji}+\nabla_j S_{ik}+\nabla_i S_{kj}=0,$$

from which and (2.1), we also find

$$(2.2)\quad \nabla_k K_{ji}-\nabla_j K_{ki}=-\phi_i^{\ r}\nabla_r S_{kj}-2S_{kj}\eta_i+(n-1)(\phi_{ki}\eta_j-\phi_{ji}\eta_k+2\phi_{kj}\eta_i).$$

Differentiating (1.1) covariantly and using (2.1), we have

$$\begin{aligned}(2.3)\quad (n+3)\nabla_t B_{kji}^{\ \ \ t}=&(n+2)(\nabla_k K_{ji}-\nabla_j K_{ki})-\phi_k^{\ r}\phi_j^{\ s}(\nabla_r K_{si}-\nabla_s K_{ri})+2\phi_i^{\ s}\phi_k^{\ r}\nabla_s K_{rj}\\ &+\eta^r(\eta_k\nabla_r K_{ji}-\eta_j\nabla_r K_{ki})-(n+2)\eta_k S_{ji}+n\eta_j S_{ki}+2(n+1)\eta_i S_{kj}\\ &+\frac{1}{n+1}(g_{ki}\eta_i-g_{ji}\eta_k)\eta^r\nabla_r K+\frac{n-1}{2(n+1)}\{(g_{ki}-\eta_k\eta_i)\nabla_j K-(g_{ji}-\eta_j\eta_i)\nabla_k K\\ &+(\phi_{ki}\phi_j^{\ r}-\phi_{ji}\phi_k^{\ r}+2\phi_{kj}\phi_i^{\ r})\nabla_r K\}+(n-1)\{(n+2)\eta_k\phi_{ji}\\ &-n\eta_j\phi_{ki}-2(n+1)\eta_i\phi_{kj}\}.\end{aligned}$$

Transvecting (2.3) with $\phi_a^{\ k}\phi_b^{\ j}$ and changing the indices $a, b$ to $j, k$ respectively in the equation thus obtained, we find by adding the resulting equation to (2.3)

$$\begin{aligned}&\nabla_t B_{kji}^{\ \ \ t}+\phi_j^{\ r}\phi_k^{\ s}\nabla_t B_{rsi}^{\ \ \ t}\\ &=(\nabla_k K_{ji}-\nabla_j K_{kj})-\phi_k^{\ r}\phi_j^{\ s}(\nabla_r K_{si}-\nabla_s K_{ri})+(n-1)(\eta_k\phi_{ji}-\eta_j\phi_{ki})-\eta_k S_{ji}+\eta_j S_{ki}\\ &+\frac{1}{2(n+3)}(g_{ki}\eta_j-g_{ji}\eta_k)\eta^t\nabla_t K.\end{aligned}$$

On the other side, using (1.2), we have

$$\phi_j^{\ r}\phi_k^{\ s}\nabla_t B_{rsi}^{\ \ \ t}=-\nabla_t B_{kji}^{\ \ \ t},$$

from which and the last equation,

$$(2.4)\quad \nabla_k K_{ji}-\nabla_j K_{ki}-\phi_k^{\ r}\phi_j^{\ s}(\nabla_r K_{si}-\nabla_s K_{ri})-\eta_k S_{ji}+\eta_j S_{ki}$$
$$+\frac{1}{2(n+3)}(g_{ki}\eta_j-g_{ji}\eta_k)\eta^r\nabla_r K+(n-1)(\eta_k\phi_{ji}-\eta_j\phi_{ki})=0.$$

Contracting the last equation with $\eta^k$ and $\eta^k g^{ji}$ respectively, we find

$$(2.5)\qquad \eta^t\nabla_t K=0,\quad \eta^t\nabla_t K_{ji}=0.$$

Substituting (2.4) and (2.5) into (2.3), we obtain

$$\frac{n+3}{n-1}\nabla_t B_{kji}^{\ \ \ t}=\nabla_k K_{ji}-\nabla_j K_{ki}-\eta_k\{S_{ji}-(n-1)\phi_{ji}\}+\eta_j\{S_{ki}-(n-1)\phi_{ki}\}$$
$$+2\eta_i\{S_{kj}-(n-1)\phi_{kj}\}+\frac{1}{2(n+1)}\{(g_{ki}-\eta_k\eta_i)\delta_j^{\ t}-(g_{ji}-\eta_j\eta_i)\delta_k^{\ t}$$
$$+\phi_{ki}\phi_j^{\ t}-\phi_{ji}\phi_k^{\ t}+2\phi_{kj}\phi_i^{\ t}\}\nabla_t K.$$

Thus, in a Sasakian manifold with vanishing C-Bochner curvature tensor, we get

$$(2.6)\quad \nabla_k K_{ji}-\nabla_j K_{ki}=\eta_k\{S_{ji}-(n-1)\phi_{ji}\}-\eta_j\{S_{ki}-(n-1)\phi_{ki}\}$$
$$-2\eta_i\{S_{kj}-(n-1)\phi_{kj}\}-\frac{1}{2(n+1)}\{(g_{ki}-\eta_k\eta_i)\delta_j^{\ t}$$
$$-(g_{ji}-\eta_j\eta_i)\delta_k^{\ t}+\phi_{ki}\phi_j^{\ t}-\phi_{ji}\phi_k^{\ t}+2\phi_{kj}\phi_i^{\ t}\}\nabla_t K,$$

$$(2.7)\quad \nabla_k S_{ji}=\eta_j K_{ki}-\eta_i K_{kj}+\frac{1}{2(n+1)}\{\phi_{jk}\delta_i^{\ t}-\phi_{ik}\delta_j^{\ t}$$
$$+2\phi_{ji}\delta_k^{\ t}+(g_{ik}-\eta_i\eta_k)\phi_j^{\ t}-(g_{jk}-\eta_j\eta_k)\phi_i^{\ t}\}\nabla_t K$$

(See also [5], [9]).

In the rest of this section, we are going to compute $\nabla_k K_{ji}$ and $\nabla_k Z_{ji}$ by using (2.5), (2.6) and (2.7).

Differentiating $S_{ji}=\phi_j^{\ t}K_{ti}$ covariantly gives

$$\nabla_k S_{ji}=(\eta_j\delta_k^{\ t}-\eta^t g_{kj})K_{ti}+\phi_j^{\ t}\nabla_k K_{ti}=\eta_j K_{ki}-(n-1)\eta_i g_{kj}+\phi_j^{\ t}\nabla_k K_{ti}$$

as already shown in (2.1), which together with (2.7) implies

$$(2.8)\quad \phi_j^{\ t}\nabla_k K_{ti}=(n-1)\eta_i g_{kj}-\eta_i K_{kj}+\frac{1}{2(n+1)}\{\phi_{jk}\delta_i^{\ t}-\phi_{ik}\delta_j^{\ t}$$
$$+2\phi_{ji}\delta_k^{\ t}+(g_{ik}-\eta_i\eta_k)\phi_j^{\ t}-(g_{jk}-\eta_j\eta_k)\phi_i^{\ t}\}\nabla_t K.$$

Transvecting (2.8) with $\phi_1^{\,j}$ and using (2.5) and (2.6) give

$$(2.9)\quad \nabla_k K_{ji} = -\eta_j\{S_{ki}-(n-1)\phi_{ki}\} - \eta_i\{S_{kj}-(n-1)\phi_{kj}\} + \frac{1}{2(n+1)}\{(g_{jk}-\eta_j\eta_k)\delta_i^{\,t} + \phi_{ik}\phi_j^{\,t} + (g_{ik}-\eta_i\eta_k)\delta_j^{\,t} + \phi_{jk}\phi_i^{\,t} + 2(g_{ji}-\eta_j\eta_i)\delta_t^{\,k}\}\nabla_t K,$$

which and (1.3) also implies

$$(2.10)\quad \nabla_k Z_{ji} = -\eta_j\Big\{S_{ki}-\Big(\frac{K}{n-1}-1\Big)\phi_{ki}\Big\} - \eta_i\Big\{S_{kj}-\Big(\frac{K}{n-1}-1\Big)\phi_{kj}\Big\} - \frac{1}{n-1}(\nabla_k K)g_{ji} + \frac{1}{n-1}(\nabla_k K)\eta_j\eta_i + \frac{1}{2(n+1)}\{(g_{jk}-\eta_j\eta_k)\delta_i^{\,t} + \phi_{ik}\phi_j^{\,t} + (g_{ik}-\eta_i\eta_k)\delta_j^{\,t} + \phi_{jk}\phi_i^{\,t} + 2(g_{ji}-\eta_j\eta_i)\delta_k^{\,t}\}\nabla_t K.$$

## 3. Laplacian $\Delta(Z_{ji}Z^{ji})$

In order to compute the Laplacian

$$(3.1)\quad \frac{1}{2}\Delta(Z_{ji}Z^{ji}) = g^{kj}(\nabla_k\nabla_j Z_{ih})Z^{ih} + (\nabla_k Z_{ji})(\nabla^k Z^{ji}),$$

where the tensor $Z_{ji}$ is defined by (1.3), in a Sasakian manifold with vanishing $C$-Bochner curvature tensor, we first consider the first term $g^{kj}(\nabla_k\nabla_j Z_{ih})Z^{ih}$ in the right hand side of (3.1).

Taking account of (2.7) and (2.10), we obtain

$$(3.2)\quad \nabla_k\nabla_j Z_{ih} = -\phi_{ki}\Big\{S_{jh}-\Big(\frac{K}{n-1}-1\Big)\phi_{jh}\Big\} - \eta_i\nabla_k\Big\{S_{jh}-\Big(\frac{K}{n-1}-1\Big)\phi_{jh}\Big\} - \phi_{kh}\Big\{S_{ji}-\Big(\frac{K}{n-1}-1\Big)\phi_{ji}\Big\} - \eta_h\nabla_k\Big\{S_{ji}-\Big(\frac{K}{n-1}-1\Big)\phi_{ji}\Big\} - \frac{1}{n-1}(\nabla_k\nabla_j K)g_{ih} + \frac{1}{n-1}(\nabla_k\nabla_j K)\eta_i\eta_h + \frac{1}{n-1}(\nabla_j K)(\phi_{ki}\eta_h+\eta_i\phi_{kh}) - \frac{1}{2(n+1)}\{(\phi_{kj}\eta_i+\eta_j\phi_{ki})\delta_h^{\,t} - (\eta_h g_{kj}-\eta_j g_{kh})\phi_i^{\,t} - \phi_{hj}(\eta_i\delta_k^{\,t}-\eta^t g_{ki}) + 2(\phi_{ki}\eta_h + \eta_i\phi_{kh})\delta_j^{\,t} + (\phi_{kh}\eta_j+\eta_h\phi_{kj})\delta_i^{\,t} - (\eta_i g_{kj}-\eta_j g_{ki})\phi_h^{\,t} - \phi_{ij}(\eta_h\delta_k^{\,t}-\eta^t g_{hk})\}\nabla_t K + \frac{1}{2(n+1)}\{(g_{ji}-\eta_j\eta_i)\delta_h^{\,t} + \phi_{hj}\phi_i^{\,t} + (g_{hj}-\eta_h\eta_j)\delta_i^{\,t} + \phi_{ij}\phi_h^{\,t} + 2(g_{ih}-\eta_i\eta_h)\delta_j^{\,t}\}\nabla_k\nabla_t K.$$

Transvecting (3.2) with $g^{kj}Z^{ih}$ and making use of $Z_{ji}\eta^i=0$ and $Z_i^{\,i}=0$, we can easily verify that

$$(3.3)\quad g^{kj}(\nabla_k\nabla_j Z_{ih})Z^{ih} = -2\phi_{si}S_h^{\,s}Z^{ih} + \frac{1}{n+1}\{Z_k^{\,t} + \phi_{hk}\phi_i^{\,t}Z^{ih}\}\nabla^k\nabla_t K.$$

On the other hand, taking account of the skew-symmetry of $S_{ji}$, we have

$$\phi_{si}S_h^{\ s}Z^{ih}=K_{ih}Z^{ih}$$
$$=K_{ih}K^{ih}-\left(\frac{K}{n-1}-1\right)K+(n-1)\left(\frac{K}{n-1}-n\right)$$

and

$$\phi_{hk}\phi_i^{\ t}Z^{ih}=Z_k^{\ t}.$$

Substituting the last two equations into (3.3) implies

$$(3.4)\quad g^{kj}(\nabla_k\nabla_jZ_{ih})Z^{ih}=-2K_{ih}K^{ih}+2\left(\frac{K}{n-1}-1\right)K$$
$$-2(n-1)\left(\frac{K}{n-1}-n\right)+\frac{2}{n+1}\{\nabla_k(Z^{kt}\nabla_tK)-(\nabla_kZ_t^{\ k})\nabla^tK\}.$$

Next we consider the second term in the right hand side of (3.1). Taking account of (1.3), we have by a straightforward computation

$$(\nabla_kZ_{ji})(\nabla^kZ^{ji})=\{\nabla_kK_{ji}-\frac{1}{n-1}(\nabla_kK)g_{ji}+\frac{1}{n-1}(\nabla_kK)\eta_j\eta_i$$
$$+\left(\frac{K}{n-1}-n\right)(\phi_{kj}\eta_i+\eta_j\phi_{ki})\}\{\nabla^kK^{ji}-\frac{1}{n-1}(\nabla^kK)g^{ji}$$
$$+\frac{1}{n-1}(\nabla^kK)\eta^j\eta^i+\left(\frac{K}{n-1}-n\right)(\phi^{kj}\eta^i+\eta^j\phi^{ki})\},$$

which reduces to

$$(3.5)\quad (\nabla_kZ_{ji})(\nabla^kZ^{ji})=(\nabla_kK_{ji})(\nabla^kK^{ji})-\frac{1}{n-1}(\nabla_kK)(\nabla^kK)$$
$$+4\left(\frac{K}{n-1}-n\right)\{-K+n(n-1)\}+2(n-1)\left(\frac{K}{n-1}-n\right)^2$$

because of $\phi_{kj}\eta_i\nabla^kK^{ji}=-K+n(n-1)$ which is a consequence of $K_{ji}\eta^i=(n-1)\eta_j$. On the other hand we can also find by using(2.9)

$$(3.6)\quad (\nabla_kK_{ji})(\nabla^kK^{ji})$$
$$=[\eta_j\{S_{ki}-(n-1)\phi_{ki}\}+\eta_i\{S_{kj}-(n-1)\phi_{kj}\}+\frac{1}{2(n+1)}\Big\{\{(-g_{jk}+\eta_j\eta_k)\delta^t_{\ i}$$
$$-\phi_{ik}\phi_j^{\ t}+(-g_{ik}+\eta_i\eta_k)\delta^{\ t}_j-\phi_{jk}\phi_i^{\ t}+2(-g_{ji}+\eta_j\eta_i)\delta^t_{\ k}\}\nabla_tK]\ [\eta^j\{S^{ki}-(n-1)\phi^{ki}\}$$
$$+\eta^i\{S^{kj}-(n-1)\phi^{kj}\}+\frac{1}{2(n+1)}\{(-g^{jk}+\eta^j\eta^k)g^{is}-\phi^{ik}\phi^{js}+(-g^{ji}+\eta^j\eta^i)g^{ks}$$
$$-\phi^{jk}\phi^{is}+2(-g^{ji}+\eta^j\eta^i)g^{ks}\}\nabla_sK]$$
$$=2K_{ji}K^{ji}-4(n-1)K+2n(n-1)^2+\frac{2}{n+1}(\nabla_tK)(\nabla^tK),$$

where we have used (2.5), $S_{ji}S^{ji}=K_{ji}K^{ji}-(n-1)^2$ and $\phi_{ji}S^{ji}=K-(n-1)$.
Finally contracting (2.10) with $g^{kj}$ and using (2.5), we get

$$(3.7) \qquad (\nabla_k Z_i^{\ k})\nabla^i K=\frac{n-3}{2(n-1)}(\nabla_t K)(\nabla^t K).$$

Substituting (3.6) and (3.7) into (3.5) and (3.4) respectively and substituting the resulting equations into (3.1), we obtain

$$(3.8) \qquad \frac{1}{2}\Delta(Z_{ji}Z^{ji})=\frac{2}{n+1}\nabla_k(Z^{kt}\nabla_t K),$$

which implies

LEMMA 1. *In a Sasakian manifold with constant scalar curvature whose C-Bochner curvature tensor vanishes identically, we have* $\Delta(Z_{ji}Z^{ji})=0$.

### 4. Proof of main theorems

In a Sasakian manifold with constant scalar curvature whose C-Bochner curvature tensor vanishes, we have

$$\frac{1}{2}\Delta(Z_{ji}Z^{ji})=g^{kj}(\nabla_k\nabla_j Z_{ih})Z^{ih}+(\nabla_k Z_{ji})(\nabla^k Z^{ji})$$
$$=g^{kj}\nabla_k[\nabla_i Z_{jh}+\eta_j\{S_{ih}-\left(\frac{K}{n-1}-1\right)\phi_{ih}\} -\eta_i\{S_{jh}-\left(\frac{K}{n-1}-1\right)\phi_{jh}\}$$
$$-2\eta_h\{S_{ji}-\left(\frac{K}{n-1}-1\right)\phi_{ji}\}]Z^{ih}+(\nabla_k Z_{ji})(\nabla^k Z^{ji})=0$$

by using (2.10), $\nabla_k K=0$ and Lemma 1. Applying the Ricci's identity to the last equation and taking account of $Z_i^{\ i}=0$ and $Z_{ji}\eta^i=0$, we can easily see that

$$(4\cdot 1) \quad K_i^{\ t}Z_{th}Z^{ih}-K_{sih}^{\ \ \ t}Z_t^{\ s}Z^{ih}-3Z_{ih}Z^{ih}+(\nabla_k Z_{ji})(\nabla^k Z^{ji})=0$$

with the help of $\nabla^t Z_{th}=0$ and $\phi_{si}S_h^{\ s}Z^{ih}=Z_{ih}Z^{ih}$. On the other hand, using (1.1) with $B_{kji}^{\ \ \ h}=0$ directly, we get

$$(4.2) \quad K_i^{\ t}Z_{th}Z^{ih}-K_{sih}^{\ \ \ t}Z_t^{\ s}Z^{ih}=\frac{n-1}{n+3}Z_i^{\ t}Z_{th}Z^{ih}$$
$$+\frac{1}{n+3}\left\{\frac{n+3}{n+1}K+\frac{2(n-1)}{n+1}+2n+2\right\} Z_{ji}Z^{ji}.$$

Also using (3.5) and (3.6), we have

$$(4.3) \qquad (\nabla_k Z_{ji})(\nabla^k Z^{ji})=2Z_{ji}Z^{ji}$$

because of

$$(4.4) \qquad Z_{ji}Z^{ji}=K_{ji}K^{ji}-\frac{1}{n-1}K^2+2K-n(n-1).$$

Substituting (4.2) and (4.3) into (4.1), we have

$$\frac{n-1}{n+3} Z_i^{\ t} Z_{th} Z^{ih} + \frac{1}{n+1} \{K+n-1\} Z_{ji} Z^{ji} = 0. \tag{4.5}$$

Now we assume that

$$K_{ji} K^{ji} \leq \frac{K^2}{n-1} \tag{4.6}$$

and then consider the following two cases:

(i) $K \leq 0$, (ii) $K > 0$

In the first case, using (4.4) and (4.5), we have

$$Z_{ji} Z^{ji} \leq 2K - n(n-1) < 0,$$

which implies

$$Z_{ji} = 0.$$

Hence, taking account of (1.3), we find

$$K_{ji} = \left(\frac{K}{n-1} - 1\right) g_{ji} - \left(\frac{K}{n-1} - n\right) \eta_j \eta_i,$$

which means that the Sasakian manifold is $C$-Einstein, and consequently locally $C$-Fubinian with the help of Theorem $A$.

In the second case we need the following lemma.

LEMMA 2. (Okumura [7]) *Let $a_i$ $(i=1, \cdots, n)$ be real numbers such that $\sum_{i=1}^{n} a_i = 0$. If we put $k^2 = \sum_{i=1}^{n} a_i^2$, i.e. $k = \sqrt{\sum_{i=1}^{n} a_i^2}$, then the inequalities*

$$-\frac{n-2}{\sqrt{n(n-1)}} k^3 \leq \sum_{i=i}^{n} a_i^3 \leq \frac{n-2}{\sqrt{n(n-1)}} k^3$$

*hold good.*

By means of Lemma 2, since $Z_i^{\ i} = 0$, (4.5) reduces to

$$0 = \frac{n-1}{n+3} Z_i^{\ t} Z_{th} Z^{ih} + \left\{\frac{K}{n+1} + \frac{n-1}{n+1}\right\} Z_{ji} Z^{ji}$$
$$\geqq \left\{\frac{K+n-1}{n+1} - \frac{(n-2)\sqrt{n-1}}{(n+3)\sqrt{n}} \sqrt{2K - n(n-1)}\right\} Z_{ji} Z^{ji},$$

from which, putting $Q = \sqrt{2K - n(n-1)}$, we have

$$0 \geq \left\{\left(\frac{Q}{\sqrt{2(n+1)}} - \frac{(n-2)\sqrt{n^2-1}}{(n+3)\sqrt{2n}}\right)^2 + \frac{2(n-1)}{n(n+1)(n+3)^2} (2n+1)(n^2+n-1)\right\} Z_{ji} Z^{ji},$$

and consequently

$$Z_{ji}=0.$$

Therefore we complete the proof of Theorem 1.

Next, we prepare a lemma in order to prove Theorem 2.

LEMMA 3. (Okumura [6]) *Let* $a_1, \cdots, a_n, b$ *be* $n+1$ $(n>1)$ *real numbers satisfying the following inequality.*

$$(\sum_{i=1}^{n} a_i)^2 \geq (n-1)\sum_{i=1}^{n}(a_i)^2+b \ (resp. >).$$

*Then, for any distinct i and j, we have*

$$2a_i a_j \geq \frac{b}{n-1} \ (resp. >).$$

First of all, replacing the quantities about $Z_{ji}$ by those of $K_{ji}$ in (3.8), we can see

$$(4.7) \quad \frac{1}{2}\nabla(K_{ji}K^{ji})=\left(\frac{2}{n+1}K_{ji}+\frac{K-(n-1)}{n-1}g_{ji}\right)\nabla^j\nabla^i K+\frac{2}{n+1}\|\nabla_j K\|^2.$$

Let $a_i$ $(i=1, \cdots, n)$ be the eigenvalues of $K_j^{\ i}$. Then the assumption $K_{ji}K^{ji}=$ const. $<\frac{K^2}{n-1}$ implies

$$a_i a_j > 0 \ (i \neq j)$$

by means of Lemma 3. On the other hand $K_{ji}\eta^i=(n-1)\eta_j$, and consequently we can assume $a_n=n-1$.

Hence $a_i>0$ $(i=1, \cdots, n)$, which means $A_{ji}=\frac{2}{n+1}K_{ji}+\frac{K-(n-1)}{n-1}g_{ji}$ is positive definite. Therefore (4.7) yields

$$A^{ji}\nabla_j\nabla_i K<0$$

for a positive definite quadratic form $A_{ji}dx^j dx^i$, where $A^{ji}=A_{ts}g^{tj}g^{si}$, Hence by means of E. Hopf's theorem [12] $K$ is constant. Therefore Theorem 1 implies Theorem 2.

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