ON pc-RINGS

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In this paper rings over which all the cyclic modules are pseudo-injective, called here as pc-rings, are studied and it is shown that

(i) A ring $R$ is left pc iff $R/A$ is left pc for each ideal $A$ of $R$.

(ii) A left pc-ring is self-pseudo-injective. Moreover, if $R$ is noetherian then $R/J$ is semi-simple artinian where $J$ is the Jacobson radical of $R$.

(iii) Factor of a pc-ring is self-pseudo-injective. Conversely, if each factor of a duo ring is self-pseudo-injective then $R$ is a pc-ring.

(iv) If a prime left Goldie ring $R$ is pc then each quotient of $R$ by a closed ideal is injective.

Throughout this paper $R$ will denote a ring with unit and modules are unitary. $J(R)$ will stand for the Jacobson radical and $Z(R)$ for singular ideal of $R$. $M \triangle N$ will mean that $M$ is an essential extension of $N$. An element $m$ of a module $M$ is said to be singular if $R \triangle (0: m)$. The module $M$ is nonsingular if none of its non-zero elements is singular. A ring is said to be left Goldie if it satisfies ascending Chain Condition on annihilator left ideals and does not contain any infinite direct sum of left ideals. An $R$-module $M$ is said to be pseudo injective if every $R$-monomorphism of each $R$-submodule of $M$ into $M$ can be extended to an $R$-endomorphism of $M$. A ring $R$ is self-pseudo injective if it is pseudo injective as an $R$-module.

**LEMMA 1.** Let $M$ be an $R$-module and let $A$ be an ideal of $R$ which annihilates $M$. Then $M$ is a pseudo-injective $R$-module iff it is pseudo-injective as an $R/A$-module.

**PROOF.** Trivial, since under the above condition, we have $\text{Hom}_R (M, M) = \text{Hom}_{R/A} (M, M)$.

**PROPOSITION 2.** If $R$ is a self pseudo-injective ring (with 1) then $J(R) = Z(R)$ and $R/J(R)$ is von Neumann regular.

**PROOF.** Suppose $E = \text{Hom}_R (R, R)$. Then since $R$ has 1, the mapping
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\[ \theta : f \rightarrow f(1) \]
of \( E \) onto \( R \) is a ring isomorphism. Under this map \( \mathcal{L} \in R \) corresponds to

\[ f : x \mapsto x \mathcal{L} \]

So, \( x \in \ker f \iff x \mathcal{L} = 0 \iff x \in (0 : \mathcal{L}) \)

Hence \( \ker f = (0 : \mathcal{L}) \).

Now, since \( R \) is a pseudo-injective \( R \)-module, we have, by [2, Theorem 4.2]:

\[ J(E) = \{ f \in E/R \triangle \ker f \} \]

Due to the isomorphism we have

\[ J(R) = \{ \mathcal{L} \in R/R \triangle (0 : \mathcal{L}) \} = Z(R) \]

Again, by the second part of the above cited theorem of [2] we know that \( E - J(E) \) is von Neumann regular. It follows that \( R/J(R) \) is von Neumann regular in view of the fact that \( \theta \) maps

\( \{ f \in E/R \triangle \ker f \} \) into \( Z(R) \)

which is shown to be \( J(R) \) because of the self-pseudo injectivity of \( R \).

**DEFINITION 1.** A ring \( R \) will be called left(right) \( pc \)-ring if every left(right) cyclic \( R \)-module is pseudo-injective. \( R \) is said to be \( pc \) if it is right and left \( pc \).

**PROPOSITION 3.** A ring \( R \) is left \( pc \) iff \( R/A \) is left \( pc \) for each two sided ideal \( A \) of \( R \).

**PROOF.** Let \( R \) be a left \( pc \) ring and \( A \) an ideal of \( R \). Let \( I/A \) be any left ideal of \( R/A \). Consider the \( R/A \)-module \( (R/A)/(I/A) \). In view of the \( R \)-isomorphism

\[ (R/A)/(I/A) \cong R/I \]

and the fact that \( I \) annihilates the module \( R/I \), \( A \) also annihilates the \( R \)-module \( R/I \). Therefore \( R/I \) may be considered as an \( R/A \)-module.

Now, \( R \) is left \( pc \Rightarrow (R/I) \) is \( R \)-pseudo-injective. But the ideal \( A \) annihilates the \( R \)-module \( (R/I) \). So, by Lemma 1, \( R/I \) considered as an \( R/A \)-module is \( R/A \)-pseudo-injective. Hence any cyclic \( R/A \)-module is \( R/A \)-pseudo-injective. \( R/A \) is thus a \( pc \)-ring.

The converse is obvious.

**PROPOSITION 4.** Any left \( pc \) ring \( R \) is self-pseudo-injective. Moreover, if \( R \) is noetherian then \( R/J(R) \) is semi-simple artinian.

**PROOF.** Since \( R^R \) is generated by the identity, it is a cyclic left \( R \)-module. \( R^R \) is therefore, self-pseudo-injective.
Next, self-pseudo-injectivity of \( R \) implies von Neumann regularity of \( R/J(R) \) (Proposition 2). Moreover, \( R \) is noetherian \( \implies R/J(R) \) is noetherian. Thus, since \( R/J(R) \) is noetherian and regular, it is semi-simple artinian.

**THEOREM 5.** Factor of a pc-ring \( R \) is self-pseudo-injective. Conversely, if each factor of a duo ring \( R \) is self-pseudo injective then \( R \) is a pc-ring.

**PROOF.** Let \( A \) be a left ideal of a pc ring \( R \). Then \( R/A \) is pc by Proposition 3 and hence self pseudo injective by Proposition 4.

Conversely, suppose that each factor ring of \( R \) is self-pseudo-injective. Let \( M \) be a cyclic \( R \)-module. Then \( M \cong R/A \) for some left ideal \( A \) of \( R \). By assumption, \( R/A \) is \( R/A \)-pseudo-injective. Hence, by Lemma 1, \( R/A \) is \( R \)-pseudo-injective. Thus \( R \) is pc.

**PROPOSITION 6.** Let \( R \) be a pc ring which is prime left Goldie. Then any quotient of \( R \) by a closed ideal is injective.

**PROOF.** \( R \) is pc \( \Rightarrow R/I \) is pseudo-injective.

Furthermore, \( R \) is prime left Goldie implies \( R \) is nonsingular.

Now, \( Z(R)=0 \) and \( I \) is closed ideal of \( R \)

\[ \Rightarrow Z(R/I)=0 \quad \text{[1, Lemma 2.3]} \]

\[ \Rightarrow R/I \text{ is torsionfree in Levy's sense [3, Lemma 4.1]} \]

Thus, \( R/I \), being a Levy-torsionfree pseudo-injective module, is injective by [3, Theorem 4.7].

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**REFERENCES**

