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CHARACTERIZATIONS OF SPACES USING T_0-IDENTIFICATION SPACES

By Charles Dorsett

1. Introduction

In this paper T_0-identification spaces are used to characterize spaces which are R_0, R_1, regular, completely regular, normal R_0, or pseudometrizable, where the T_1 axiom is not included in the definitions of regular, completely regular, and normal. Regular T_1, completely regular T_1, and normal T_1 will be denoted by T_3, $T_{3\frac{1}{2}}$, and T_4, respectively. Listed below are definitions and theorems that will be utilized in this paper.

DEFINITION 1.1. A topological space (X,T) is R_0 if and only if for each closed set C and for each $x \notin C$, $C \cap \overline{\{x\}} = \phi$ [1].

THEOREM 1.1. *The following are equivalent*: (a) (X,T) *is* R_0, (b) *if* $O \in T$ *and* $x \in O$, *then* $\overline{\{x\}} \subset O$, *and* (c) *for* $x, y \in X$, *either* $\overline{\{x\}} = \overline{\{y\}}$ *or* $\overline{\{x\}} \cap \overline{\{y\}} = \phi$ [1].

DEFINITION 1.2. If $\sim$ is an equivalence relation on (X, T), then the $\sim$ identification space of (X,T) is $(\mathscr{D}\sim, \mathscr{Q}\sim)$, where $\mathscr{D}\sim$ is the set of equivalence classes of $\sim$ and $\mathscr{Q}\sim$ is the decomposition topology on $\mathscr{D}\sim$ [3].

DEFINITION 1.3. Let $\sim^\circ$ be the equivalence relation on (X,T) defined by $x \sim^\circ y$ if and only if $\overline{\{x\}} = \overline{\{y\}}$. The T_0-identification space of (X, T) is the $\sim^\circ$ identification of (X, T), which is T_0. Let $X_0 = \mathscr{D}\sim^\circ$, let $S_0 = \mathscr{Q}\sim^\circ$, and let $P : (X, T) \rightarrow (X_0, S_0)$ be the natural map [3].

DEFINITION 1.4. For a space (X,T) let $\sim'$ be the relation in $X \times X$ defined by $x \sim' y$ if and only if $x \in \overline{\{y\}}$. Then $\sim'$ is not always an equivalence relation on X and $\sim^\circ \subset \sim'$.

DEFINITION 1.5. A space (X, T) is R_1 if and only if for $x, y \in X$ such that $\overline{\{x\}} \neq \overline{\{y\}}$ there exist disjoint open sets U and V such that $\overline{\{x\}} \subset U$ and $\overline{\{y\}} \subset V$ [1].

THEOREM 1.2. *A space is* T_1 *if and only if it is* R_0 *and* T_0 *and a space is* T_2

if and only if it is R_1 *and* T_0 [1].

THEOREM 1.3. *A space* (X, T) *is* R_1 *if and only if* (X_0, S_0) *is* T_2 [4].

2. Characterizations

THEOREM 2.1. *The following are equivalent*: (a) (X, T) *is* R_0, (b) $\sim'$ *is an equivalence relation on* X, (c) $(\mathscr{D}\sim', \mathscr{O}\sim')=(X_0, S_0)$, $X_0=\{\overline{\{x\}}\,|\,x\in X\}$, (X_0, S_0) *is* T_1, P *is closed open, and* $P^{-1}(P(O))=O$ *for all* $O\in T$, (d) (X_0, S_0) *is* R_0, *and* (e) (X_0, S_0) *is* T_1.

PROOF. (a) implies (b): If $(x,y)\in\sim'$, then $x\in\overline{\{y\}}$ and $\overline{\{x\}}\cap\overline{\{y\}}\neq\phi$, which implies $\overline{\{x\}}=\overline{\{y\}}$ and $(x,y)\in\sim^\circ$. Hence $\sim'=\sim^\circ$, which is an equivalence relation.

(b) implies (c): If $x\in X$ and C_x is the equivalence class of $\sim'$ containing x, then $C_x=\overline{\{x\}}$. Thus $\{\overline{\{x\}}\,|\,x\in X\}$ is a decomposition of X, which implies (X, T) is R_0. By the argument above $\sim'=\sim^\circ$, and thus $(\mathscr{D}\sim', \mathscr{O}\sim')=(X_0, S_0)$ and $X_0=\{\overline{\{x\}}\,|\,x\in X\}$. If $O\in T$, then $P^{-1}(P(O))=\bigcup_{x\in O}\overline{\{x\}}=O$, which implies $P(O)\in S_0$, and thus P is open. For C closed in X, $P(C)=X_0\backslash P(X\backslash C)$, which is closed, and thus P is closed. If $\overline{\{x\}}\in X_0$, then $\overline{\{\overline{\{x\}}\}}=P(\overline{\{x\}})=P(\overline{\{x\}})=\{\overline{\{x\}}\}$, which implies (X_0, S_0) is T_1.

(c) implies (d): Since every T_1 space is R_0, then (X_0, S_0) is R_0.

(d) implies (e): Since (X_0, S_0) is R_0 and T_0, then (X_0, S_0) is T_1.

(e) implies (a): Let $x\in X$ and let C_x be the equivalence class of $\sim^\circ$ containing x. Then $C_x\subset\overline{\{x\}}$. Since $\{C_x\}$ is closed in X_0, then $x\in C_x=P^{-1}(\{C_x\})$, which is closed, and thus $\overline{\{x\}}\subset C_x$ and $C_x=\overline{\{x\}}$. Hence $X_0=\{\overline{\{x\}}\,|\,x\in X\}$, which implies (X, T) is R_0.

The following corollary can be obtained by using Theorem 1.3 and Theorem 1.2.

COROLLARY 2.1. *A space* (X,T) *is* R_1 *if and only if* (X_0, S_0) *is* R_1.

Note that if (X, T) *is* R_0, *then* $\overline{\{x\}}$ *compact for all* $x\in X$.

THEOREM 2.2. *The following are equivalent*: (a) (X, T) *is regular*, (b) (X_0, S_0) *is regular, and* (c) (X_0, S_0) *is* T_3.

PROOF. (a) implies (b): Since (X, T) is regular, then (X, T) is R_0. By Theorem 2.1, (X_0, S_0) is an upper semi-continuous decomposition of X into compact sets and since (X, T) is regular, then (X_0, S_0) is regular [2].

(b) implies (c): Since (X_0, S_0) is regular T_0, then (X_0, S_0) is T_3.

(c) implies (a): By Theorem 2.1, $X_0=\{\overline{\{x\}}\,|\,x\in X\}$, P is open, and $P^{-1}(P(O))=O$ for all $O\in T$. Let $O\in T$ and let $x\in O$. Then $\overline{\{x\}}\in P(O)$, which is open in X_0. Thus there exists an open set $\mathscr{V}$ such that $\overline{\{x\}}\in\mathscr{V}\subset\overline{\mathscr{V}}\subset P(O)$, which implies $x\in P^{-1}(\mathscr{V})\subset P^{-1}(\overline{\mathscr{V}})\subset O$. Hence (X, T) is regular.

THEOREM 2.3. *The following are equivalent*: (a) (X, T) *is completely regular*, (b) (X_0, S_0) *is completely regular, and* (c) (X_0, S_0) *is* $T_{3\frac{1}{2}}$.

PROOF (a) implies (b): Since (X, T) is completely regular, then (X, T) is R_0. By Theorem 2.1, $X_0=\{\overline{\{x\}}\,|\,x\in X\}$ and P is open. Let $\mathscr{C}$ be closed in X_0 and let $\overline{\{x\}}\notin\mathscr{C}$. Then $P^{-1}(\mathscr{C})$ is closed in X and $x\notin P^{-1}(\mathscr{C})$ and there exists a continuous function $f: (X, T)\to[0,1]$ such that $f(x)=0$ and $f(P^{-1}(\mathscr{C}))=1$. Let $g: (X_0, S_0)\to[0,1]$ defined by $g(\overline{\{y\}})=f(y)$. Then $g(\overline{\{x\}})=0$ and $g(\mathscr{C})=1$. If O is open in $[0,1]$, then $f^{-1}(O)$ is open in X and $g^{-1}(O)=P(f^{-1}(O))$ is open in X_0, which implies g is continuous. Hence, (X_0, S_0) is completely regular.

(b) implies (c): Since (X_0, S_0) is completely regular T_0, then (X_0, S_0) is $T_{3\frac{1}{2}}$.

(c) implies (a): By Theorem 2.1, $X_0=\{\overline{\{x\}}\,|\,x\in X\}$ and P is closed. Let C be closed in X and let $x\notin C$. Then $P(C)$ is closed in X_0 and $P(x)\notin P(C)$ and there exists a continuous function $f: (X_0, S_0)\to[0,1]$ such that $f(P(x))=0$ and $f(P(C))=1$. Then $f\circ P: (X, T)\to[0,1]$ is continuous and $(f\circ P)(x)=0$ and $(f\circ P)(C)=1$. Hence (X, T) is comletely regular.

THEOREM 2.4. *The following are equivalent*: (a) (X, T) *is normal* R_0, (b) (X_0, S_0) *is normal* R_0, *and* (c) (X_0, S_0) *is* T_4.

PROOF. (a) implies (b): By Theorem 2.1, (X_0, S_0) is an R_0 upper semi-continuous decomposition of X into compact sets and since (X, T) is normal, then (X_0, S_0) is normal.

(b) implies (c): Since (X_0, S_0) is nomal, R_0, and T_0, then (X_0, S_0) is T_4.

(c) implies (a): By Theorem 2.1, (X, T) is R_0, $X_0=\{\overline{\{x\}}\,[x\in X\}$, and P is closed. Let C_1 and C_2 be disjoint closed sets in X. Then $P(C_1)$ and $P(C_2)$ are disjoint closed sets in X_0 and there exist disjoint open sets $\mathscr{U}$ and $\mathscr{V}$ such that

$P(C_1)\subset\mathscr{U}$ and $P(C_2)\subset\mathscr{V}$. Then $C_1\subset P^{-1}(\mathscr{U})\in T$, and $C_2\subset P^{-1}(\mathscr{V})\in T$, and $P^{-1}(\mathscr{U})\cap P^{-1}(\mathscr{V})=\phi$. Hence, (X, T) is normal.

THEOREM 2.5. *A space (X, T) is pseudometrizable if and only if (X_0, S_0) is metrizable.*

PROOF. Suppose (X, T) is pseudometrizable. Let d be a pseudometric on X compatible with T. Since (X, T) is pseudometizable, then (X, T) is R_0 and $X_0 = \{\overline{\{x\}} \mid x\in X\}$. Let d_0 be the metric on X_0 defined by $d_0(\overline{\{x\}}, \overline{\{y\}})=d(x, y)$. Since P is continuous, open, and onto, then d_0 is compatible with S_0, which implies (X_0, S_0) is metrizable.

Conversely, suppose (X_0, S_0) is metrizable. Then (X_0, S_0) is T_1, which implies (X, T) is R_0 and $X_0=\{\overline{\{x\}} [x\in X\}$. Let P_0 be a metric on X_0 compatible with S_0 and let P be the pseudometric on X defined by $P(x, y)=P_0(\overline{\{x\}}, \overline{\{y\}})$. Since P is continuous, open, and $P^{-1}(P(O))=O$ for all $O\in T$, then P is compatible with T, which implies (X, T) is pseudometrizable.

THEOREM 2.6. *Let (X, T) be an R_0 space. Then* (a) *(X, T) is separable if and only if (X_0, S_0) is separable, and* (b) *(X, T) is second countable if and only if (X_0, S_0) is second countable.*

PROOF. By Theorem 2.1, (X_0, S_0) is T_1, $X_0=\{\overline{\{x\}} \mid x\in X\}$, P is closed open, and $P^{-1}(P(O))=O$ for all $O\in T$.

(a) Suppose (X, T) is separable. Thus (X, T) has a countable dense subset $\{x_i\}_{i=1}^{\infty}$. Then $\{\overline{\{x_i\}}\}_{i=1}^{\infty}$ is a countable dense subset of (X_0, S_0), which implies (X_0, S_0) is separable.

Conversely, suppose (X_0, S_0) is separable. Thus (X_0, S_0) has a countable dense subset $\{\overline{\{x_i\}}\}_{i=1}^{\infty}$. Then $\{x_i\}_{i=1}^{\infty}$ is a countable dense subset of X, which implies (X, T) is separable.

(b) Suppose (X, T) is second countable. Then (X_0, S_0) is an upper semi-continuous decomposition of (X, T) into compact sets and (X, T) is second countable, which implies (X_0, S_0) is second countable [2].

Conversely, suppose (X_0, S_0) is second countable. Then (X_0, S_0) has a countable base $\{\mathscr{O}_i\}_{i=1}^{\infty}$. For each $i\in N$, let $O_i=P^{-1}(\mathscr{O}_i)$. Then $\{O_i\}_{i=1}^{\infty}$ is a countable base for (X, T), which implies (X, T) is second countable.

The next corollary can be obtained by using Theorem 2.5, Theorem 2.2, Theorem 2.6, and Urysohn's Metrization Theorem.

COROLLARY 2.2. *The following are equivalent*: (a) (X, T) *is regular second countable,* (b) (X_0, S_0) *is a separable metric space, and* (c) (X, T) *is a separable pseudometric space.*

North Texas State University
Denton, Texas 76203

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