Kyungpook Math. J.
Volume 17, Number 2
December, 1977

STIELTJES TRANSFORM OF FUNCTIONS SATISFYING THE LIPSCHITZ CONDITION*

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1. Introduction

This paper will be concerned about the Stieltjes transform of a function $\phi(t)$, $S_I[\phi,x]$ defined by

$$S_I[\phi,x]=f(x)=\int_0^\infty \frac{1}{x+t}\phi(t)dt \tag{1.1}$$

where $\phi(t)\in L_1(0,R)$ for all $R>0$ and the integral exists in (1.1).

We define two operators in the following manner (see Widder, p.345):

$$I[f,K;t]=\frac{(-1)^{K-1}}{K!(K-2)!}\frac{d^{2K-1}}{dt^{2K-1}}[t^K f(t)]-\phi(x)$$

$$J[f,K;t]=\frac{d}{dt}\left[\frac{(-1)^{K-1}}{K!(K-2)!}\frac{d^{2K-1}}{dt^{2K-1}}\{t^K f(t)\}\right].$$

Our main aim is to show that the asymptotic behavior $J[f,K;t]=0(1)$ uniformly in some interval of t, implies that $\phi(t)$ satisfies the Lipschitz condition there. Also, we state a theorem about the asymptotic behavior of $I[f,K;t]$, whose proof we include here because of the completeness, although it's proof is based along the same lines as the proof of a theorem in [2].

THEOREM 1.1 *Let* $f(x)=S_I[\phi,x]$, $t>0$. *Then*

$$\int_t^{t+h}[\phi(t+y)-\phi(t)]dy=o(h) \text{ as } h\to 0 \tag{1.2}$$

implies $I[f,K;t]=o(1)$ *as* $K\to\infty$.

PROOF. It can be easily seen that

$$L_{K,t}[f(x)]=\frac{(2K-1)!}{K!(K-2)!}\int_0^\infty \frac{t^{K-1}u^K}{(t+u)^{2K}}\phi(u)du$$

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* Abstract has been published in the Notices of the A.M.S., February, 1974.
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where

$$L_{K,t}[f(x)]=\frac{(-1)^{K-1}}{K!(K-2)!}\frac{d^{2K-1}}{dt^{2K-1}}[t^K f(t)],\quad (K=2,3,\cdots).$$

Set

$$\alpha(x)=\int_1^x \frac{\phi(t)}{t+1}dt,\quad (1\le x<\infty),$$

then since (1.1) converges we get $\lim_{X\to\infty}\alpha(x)=A<\infty$.

Then

$$\int_1^\infty \frac{\phi(t)}{t}dt=\int_1^\infty \frac{t+1}{t}d\alpha(t)$$

$$=\alpha(\infty)+\int_1^\infty \frac{\alpha(t)}{t^2}dt,$$

provided the integral in the second term above converges. But it is clearly absolutely convergent. Also since $\phi(t)\in L$ in $0\le t\le 1$, therefore we have the following;

$$\int_1^\infty t^C\phi(t)dt<\infty,\quad (C=-1) \tag{1.3}$$

$$\int_0^1 t^{C'}\phi(t)dt<\infty,\quad (C'=0) \tag{1.4}$$

Because of (1.2), (1.3) and (1.4) we know that Theorem 8C of [2] gives

$$\frac{(2K-1)!}{K!(K-2)!}\int_{0+}^\infty \frac{t^{K-1}u^K}{(t+u)^{2K}}\phi(u)du-\phi(t)=o(1)\qquad as\ K\to\infty.$$

From which we get

$$I(f,K;t)=o(1)\ as\ K\to\infty.$$

2. For the stieltjes transform $f(x)=S_I[\phi,x]$, ϕ is called *the determining function and* $f(x)$ *the generating of the transform.* Main theorem in this section will be stated as follows:

THEOREM 2.1. *Suppose* $f(x)=S_I[\phi,x]$ *and let*

$$|J[f,K;t]|\le K \tag{2.1}$$

for $t\in(a,b)$ *and for some* $K>0$; *then there exists a function* $\Psi(t)$ *which is equal to* $\phi(t)$ *in* $L_1[a,b]$ *norm such that*

$$|\Psi(t_1)-\Psi(t_2)|\le K|t_1-t_2|,\ t_i\in(a,b),\ i=1,2,$$

We shall use the following representation theorem for the Stieltjes transforms as given in ([2], p. 375) for the proof of Theorem 2.1. Also in the following $L_{K,t}[f]$ stands for the operator

$$L_{K,t}[f]=\frac{(-1)^{K-1}}{K!(K-2)!}\frac{d^{2K-1}}{dt^{2K-1}}[t^K f(t)],\quad (K=2, 3, \cdots)$$

THEROEM 2.2 *Necessary and sufficient conditions for $f(x)$ to be a Stieltjes transform of $\phi(t)$ with $\phi(t)\in L_1(0, R)$ for all $R>0$ are the following*:

$$(1)\qquad \int_0^\infty |L_{K,t}[f(x)]|\,dt<\infty,\quad (K=1, 2, \cdots)$$

$$(2)\qquad \lim_{K,l\to\infty}\int_0^\infty |L_{K,t}[f(x)]-L_{l,t}[f(x)]|\,dt=0$$

$$(3)\qquad \lim_{X\to 0+} xf(x)=0.$$

PROOF of Theorem 2.1. By Theorem 2.2, $L_{K,t}[f]$ converges to $\phi(t)$ in $L_1(0, R)$ (for every $R>0$) and hence in $L_1[a, b]$. Which implies that $L_{K,t}[f]$ converges in measure to $\phi(t)$ in (a, b). Hence there exists a subsequence of $L_{K,t}[f]$ say $L_{K(m),t}[f]$ which converges to $\phi(t)$ almost everywhere in (a, b). We will show that the sequence $L_{K(m),t}[f]$ converges point-wise in (a, b) to $\Psi(t)$ which obviously is equal to $\phi(t)$ in $L_1(a, b)$.

On the contrary suppose that $L_{K(m),t}[f]$ does not converge at t_0, then since $L_{K(m),t}[f]$ converges to $\phi(t)$ *a.e.*, in (a, b), there exists a sequence $t_n\in(a, b)$, such that $t_n\to t_0$ and $\lim_{m\to\infty} L_{K(m),t_n}[f]$ exists; we will prove that $\lim_{n\to\infty}\lim_{m\to\infty} L_{K(m),t_n}[f]$ exists and is equal to $\lim_{m\to\infty} L_{K(m),t_0}[f]$ (which also exists). To show that $\lim_{m\to\infty} L_{K(m),t_n}[f]$ is a Cauchy sequence in n, we apply the mean value theorem as follows:

$$\begin{aligned} &L_{K,t_{n(1)}}[f]-L_{K,t_{n(2)}}[f]\\ &\quad=(t_{n(1)}-t_{n(2)})\left[\frac{d}{dt}\frac{(-1)^{K-1}}{K!(K-2)!}\frac{d^{2K-1}}{dt^{2K-1}}\{t^K f(t)\}\right]_{t=\xi}\\ &\quad=(t_{n(1)}-t_{n(2)})J[f, K; \xi], \end{aligned}\tag{2.2}$$

where ξ is between $t_{n(1)}$ and $t_{n(2)}$.

Formula (2.2) is valid for all K and therefore by using(2.1)

$$|L_{K(m),t_{n(1)}}[f]-L_{K(m),t_{n(2)}}[f]|\leq K|t_{n(1)}-t_{n(2)}|,\tag{2.3}$$

and hence,

$$|\lim_{m\to\infty} L_{K(m),t_{n(1)}}[f]-\lim_{m\to\infty} L_{K(m),t_{n(2)}}[f]|\leq K|t_{n(1)}-t_{n(2)}|.$$

Inequality (2.4) implies that $\lim_{m\to\infty} L_{K(m),t_n}$ is a Cauchy sequence in n.

$$\text{Define } \lim_{m\to\infty}\lim_{n\to\infty} L_{K(m),t_n}[f]=M.$$

Since one can replace $t_{n(1)}$ by t_n and $t_{n(2)}$ by t_0 in (2.3) it is easy to observe that $\lim_{m\to\infty} L_{K(m),t_0}[f]=M$.

Now since every point in (a,b) can replace $t_{n(1)}$ and $t_{n(2)}$ in (2·3) and (2.4) and therefore as a consequence of the above estimates we get (2.1).

REMARK 1. The results obtained above have been motivated by the work of Ditzian ([1]) for the Laplace transforms.

REMARK 2. Some estimates for the operator $I[f, K; t]$ can be obtained when $\phi(t)$ satisfies the generalized Lipschitz condition of order γ, $(0<\gamma<1)$, *i.e.*, $\int_t^{t+h}[\phi(t+y)-\phi(t)]\,dy=0(h^{1+\gamma})$; we propose to deal with that situation in a forthcoming work. Note that in Theorem 1.1, γ was set to be equal to zero.

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