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ON THE H-FUNCTION OF n-VARIABLES

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1. Definition of the H-function of n-variables.

Following [13, p.256], the H-function of n variables is defined in terms of a multiple Mellin-Barnes type integrals in the following manner.

$$H^{A,(M_n:N_n)}_{C,D,(P_n:Q_n)}\left[\begin{matrix}x_1\\ \vdots\\ x_n\end{matrix}\left|\begin{matrix}(c_j^*,\gamma_j^{(r)});(d_j,\delta_j^{(r)})\\ (a_j^{(r)},\alpha_j^{(r)});(b_j^{(r)},\beta_j^{(r)})\end{matrix}\right.\right]$$

$$=H^{A,(M_n;N_n)}_{C,D,(P_n:Q_n)}\left[\begin{matrix}x_1\\ \\ \vdots\\ \\ x_n\end{matrix}\left|\begin{matrix}(c_1^*,\gamma_1^{(1)},\cdots,\gamma_1^{(n)}),\cdots,(c_c^*,\gamma_c^{(1)},\cdots,\gamma_c^{(n)});\\ (d_1,\delta_1^{(1)},\cdots,\delta_1^{(n)}),\cdots,(d_D,\delta_D^{(1)},\cdots,\delta_D^{(n)})\\ (a_{P_1}^{(1)},\alpha_{P_1}^{(1)}),\cdots,(a_{P_n}^{(n)},\alpha_{P_n}^{(n)});\\ (b_{Q_1}^{(1)},\beta_{Q_1}^{(1)}),\cdots,(b_{Q_n}^{(n)},\beta_{Q_n}^{(n)})\end{matrix}\right.\right]$$

$$(1.1)\quad =\frac{1}{(2\pi i)^n}\int_{-i\infty}^{i\infty}\cdots(n)\int_{-i\infty}^{i\infty} x_1^*\Big(\sum_{r=1}^{n} s_r\Big)\prod_{r=1}^{n}\{x_2^*(s_r)(x_r)^{s_r}(ds_r)\}^*$$

where

$$(1.2)\quad x_1^*(z)=\frac{\prod_{j=1}^{A}\Gamma(c_j^*+z\gamma_j^{(r)})}{\prod_{j=A+1}^{C}\Gamma(1-c_j^*-z\gamma_j^{(r)})\prod_{j=1}^{D}\Gamma(d_j+z\delta_j^{(r)})};$$

$$(1.3)\quad x_2^*(z)=\left\{\frac{\prod_{j=1}^{M_r}\Gamma(b_j^{(r)}-z\beta_j^{(r)})\prod_{j=1}^{N_r}\Gamma(1-a_j^{(r)}+\alpha_j^{(r)}z)}{\prod_{j=M_r+1}^{Q_r}\Gamma(1-b_j^{(r)}+z\beta_j^{(r)})\prod_{j=N_r+1}^{P_r}\Gamma(a_j^{(r)}-z\alpha_j^{(r)})}\right\};$$

$x_j\neq 0$ $(j=1,\cdots,n)$ and an empty product is interpreted as unity.

Further $A, C, D;\ M_1,\cdots,M_n;\ N_1,\cdots,N_n;\ P_1,\cdots,P_n$ and $Q_1,\cdots,Q_n$ are nonnegative integers, satisfying the inequalities

* The definition considered here is a slight variant of the earlier definition and can be obtained by making slight changes in the parameters and the variables but in essence the function remains the same.

$0 \leq A \leq C$, $1 \leq M_j \leq Q_j$, $0 \leq N_j \leq P_j$ $(j=1, \cdots, n)$; $a_j^{(r)}$'s, $b_j^{(r)}$'s, c_j^*'s and d_j's are all complex numbers and α's, β's, γ's and δ's are all positive numbers.

The sequence of parameters in the integrand of (1.1) are such that none of the poles coincide. That is the poles of the integrand of (1.1) are simple. In case some of the poles coincide, then by following the method of Frobenius, the integral in (1.1) can be evaluated in terms of Psi-functions and generalized Zeta functions. In this connection the reader is referred to the work of Mathai and Saxena [10].

The paths of integration, are indented, if necessary, to ensure that all the poles of $\Gamma(b_j^{(r)} - \sum_{r=1}^{n} s_r \beta_j^{(r)})$ for $j=1, \cdots, M_r$ $(r=1, \cdots, n)$ are separated from the poles of $\Gamma(c_j^* + \sum_{r=1}^{n} s_r \gamma_j^{(r)})$ for $j=1, \cdots, A$ and $\Gamma(1-a_j^{(r)} + \sum_{r=1}^{n} s_r \alpha_j^{(r)})$ for $j=1, \cdots, N_r$ $(r=1, \cdots, n)$.

The function represented by the integral (1.1) is an analytic function of $x_1, \cdots, x_n$ provided that

$$(1.4) \quad \lambda_r = \sum_{j=1}^{c} \gamma_j^{(r)} + \sum_{j=1}^{P_r} \alpha_j^{(r)} - \sum_{j=1}^{D} \delta_j^{(r)} - \sum_{j=1}^{Q_r} \beta_j^{(r)} < 0, \quad \text{for } r=1, \cdots, n.$$

From the asymptotic expansion of the gamma function it readily follows that the integral in (1.1) converges absolutely if

$$|\arg x_r| < \frac{1}{2} \pi \mu_r \quad (r=1, \cdots, n),$$

where

$$(1.5) \quad \mu_r \equiv \sum_{j=1}^{A} \gamma_j^{(r)} - \sum_{j=A+1}^{c} \gamma_j^{(r)} - \sum_{j=1}^{D} \delta_j^{(r)} + \sum_{j=1}^{N_r} \alpha_j^{(r)} + \sum_{j=1}^{M_r} \beta_j^{(r)} - \sum_{j=M_r+1}^{Q_r} \beta_j^{(r)} - \sum_{j=N_r+1}^{P_r} \alpha_j^{(r)} > 0.$$

Whenever, there is no risk of ambiguity, the H-function of n variables will be denoted by any one of the following notations

$$H_{C,D,(P_n:Q_n)}^{A,(M_n:N_n)} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad H \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

2. Identities

The following results readily follows as a consequence of the definition (1.1).

$$(2.1) \quad H_{C,D,(P_n:Q_n)}^{A,(M_n:N_n)} \left[\begin{matrix} x_1^{\mu_1} \\ \vdots \\ x_n^{\mu_n} \end{matrix} \middle| \begin{matrix} (c_j^*, \gamma_j^{(r)}) ; (d_j, \delta_j^{(r)}) \\ \\ (a_j^{(r)}, \alpha_j^{(r)}) ; (b_j^{(r)}, \beta_j^{(r)}) \end{matrix} \right]$$

$$=\frac{1}{\mu_1\mu_2\cdots\mu_n}H^{A,(M_n:N_n)}_{C,D,(P_n:Q_n)}\left[\begin{array}{c|c} x_1 & (c_j^*,\gamma_j^{(r)}/\mu_j):(d_j,\delta_j^{(r)}/\mu_j) \\ \vdots & \\ x_n & (a_j^{(r)},\alpha_j^{(r)}/\mu_j):(b_j^{(r)},\beta_j^{(r)}/\mu_j)\end{array}\right]$$

provided that $\mu_1, \mu_2, \cdots, \mu_n>0$.

(2.2) $$H^{O,(M_n:N_n)}_{C,D,(P_n:Q_n)}\left[\begin{array}{c|c} x_1^{-1} & (c_j^*,r_j^{(r)}):(d_j,\delta_j^{(r)}) \\ \vdots & \\ x_n^{-1} & (a_j^{(r)},\alpha_j^{(r)}):(b_j^{(r)},\beta_j^{(r)})\end{array}\right]$$

$$=H^{O,(N_n:M_n)}_{D,C,(Q_n:P_n)}\left[\begin{array}{c|c} x_1 & (1-d_j,\delta_j^{(r)}):(1-c_j^*,r_j^{(r)}) \\ \cdots & \\ x_n & (1-b_j^{(r)},\beta_j^{(r)}):(1-a_j^{(r)},\alpha_j^{(r)})\end{array}\right]$$

(2.3) $$H^{O,(M_n:N_n)}_{O,O,(P_n:Q_n)}\left[\begin{array}{c|c} x_1 & -:- \\ \vdots & \\ x_n & (a_j^{(r)},\alpha_j^{(r)}):(b_j^{(r)},\beta_j^{(r)})\end{array}\right]$$

$$=\prod_{j=1}^{n}H^{M_j,N_j}_{P_j,Q_j}\left[x_j\left|\begin{array}{c}(a_{p_j}^{(j)},\alpha_{p_j}^{(j)})\\(b_{Q_j}^{(j)},\beta_{Q_j}^{(j)})\end{array}\right.\right]$$

where $H^{M_j,N_j}_{P_j,Q_j}(x)$ is the Fox's H-function

For $n=2$, the H-function of n variables reduces to H-function of two variables studied by Munot and Kalla [12], Verma [17] and Mittal and Gupta [11].

3. Relations between Lauricella functions and H-function of n variables

In this section the poles of the gamma functions appearing in the various n-tuple integrals are assumed to be simple.

(3.1) $$\frac{\Gamma(a)\Gamma(b_1)\cdots\Gamma(b_n)}{\Gamma(c_1)\cdots\Gamma(c_n)}F_A^{(n)}(a,b_1,\cdots,b_n;c_1,\cdots,c_n;x_1,\cdots,x_n)$$

$$=\frac{1}{(2\pi i)^n}\int_{-i\infty}^{i\infty}\cdots(n)\cdots\int_{-i\infty}^{i\infty}\frac{\Gamma(a+s_1+\cdots+s_n)\Gamma(b_1+s_1)\cdots\Gamma(b_n+s_n)}{\Gamma(c_1+s_1)\cdots\Gamma(c_n+s_n)}$$

$$\times\Gamma(-s_1)\cdots\Gamma(-s_n)(-x_1)^{s_1}\cdots(-x_n)^{s_n}ds_1\cdots ds_n.$$

$$=H^{1,(1,\cdots,1:1,\cdots,1)}_{1,0,(1,\cdots,1:2,\cdots,2)}\left[\begin{array}{c|l} -x_1 & (a,1,\cdots,1):- \\ \vdots & (1-b_1,1),\cdots,(1-b_n,1): \\ -x_n & (0,1),\cdots,(0,1),(1-c_1,1),\cdots,(1-c_n,1)\end{array}\right].$$

(3.2) $$\frac{\Gamma(a_1)\cdots\Gamma(a_n)\Gamma(b_1)\cdots\Gamma(b_n)}{\Gamma(c)}F_B^{(n)}(a_1,\cdots,a_n;b_1,\cdots,b_n;c;x_1,\cdots,x_n)$$

$$=\frac{1}{(2\pi i)^n}\int_{-i\infty}^{i\infty}\cdots(n)\cdots\int_{-i\infty}^{i\infty}\frac{\Gamma(a_1+s_1)\cdots\Gamma(a_n+s_n)\Gamma(b_1+s_1)\cdots\Gamma(b_n+s_n)}{\Gamma(c+s_1+\cdots+s_n)}$$

$$\times\Gamma(-s_1)\cdots\Gamma(-s_n)(-x_1)^{s_1}\cdots(-x_n)^{s_n}ds_1\cdots ds_n$$

$$=H^{0,(1,\cdots,1:2,\cdots,2)}_{0,1,(2,\cdots,2:1,\cdots,1)}\left[\begin{matrix}-x_1\\ \vdots\\ -x_n\end{matrix}\middle|\begin{matrix}-\,;(c,1,\cdots,1)\\ (1-a_1,1),\ (1-b_1,1),\ \cdots,\ (1-a_n,1),\ (1-b_n,1)\\ ;(0,1),\ \cdots,\ (0,1)\end{matrix}\right]$$

(3.3) $$\frac{\Gamma(a)\Gamma(b)}{\Gamma(c_1)\cdots\Gamma(c_n)}F_C^{(n)}(a,\ b\ ;c_1,\ \cdots,\ c_n\ ;x_1,\ \cdots,\ x_n)$$

$$=\frac{1}{(2\pi i)^n}\int_{-i\infty}^{i\infty}\cdots(n)\cdots\int_{-i\infty}^{i\infty}\frac{\Gamma(a+s_1+\cdots+s_n)\Gamma(b+s_1+\cdots+s_n)}{\Gamma(c_1+s_1)\cdots\Gamma(c_n+s_n)}$$

$$\times\Gamma(-s_1)\cdots\Gamma(-s_n)(-x_n)^{s_1}\cdots(-x_n)^{s_n}ds_1\cdots ds_n$$

$$=H^{2,(1,\cdots1:0,\cdots,0)}_{2,0,(0,\cdots,0:1,\cdots,1)}\left[\begin{matrix}-x_1\\ \vdots\\ -x_n\end{matrix}\middle|\begin{matrix}(a,\ 1,\ \cdots,\ 1),\ (b,\ 1,\ \cdots,\ 1)\ ;-\\ -\ ;\\ (0,1),\ \cdots,\ (0,1),\ (1-c_1,\ 1),\ \cdots,\ (1-c_n,\ 1)\end{matrix}\right]$$

(3.4) $$\frac{\Gamma(a)\Gamma(b_1)\cdots\Gamma(b_n)}{\Gamma(c)}F_D^{(n)}(a,b_1,\ \cdots,b_n\ ;c\ ;x_1,\ \cdots,\ x_n)$$

$$=\frac{1}{(2\pi i)^n}\int_{-i\infty}^{i\infty}\cdots(n)\cdots\int_{-i\infty}^{i\infty}\frac{\Gamma(a+s_1+\cdots+s_n)\Gamma(b_1+s_1)\cdots\Gamma(b_n+s_n)}{\Gamma(c+s_1+\cdots+s_n)}$$

$$\times\Gamma(-s_1)\cdots\Gamma(-s_n)(-x_1)^{s_1}\cdots(-x_n)^{s_n}ds_1\cdots ds_n.$$

$$=H^{1,(1,\cdots1:1,\cdots,1)}_{1,1,(1,\cdots,1:1,\cdots,1)}\left[\begin{matrix}-x_1\\ \vdots\\ -x_n\end{matrix}\middle|\begin{matrix}(a,\ 1,\ \cdots,\ 1)\ ;(c,\ 1,\ \cdots,\ 1)\\ (1-b_1,\ 1),\ \cdots,\ (1-b_n,\ 1)\ ;\\ (0,\ 1),\ \cdots,\ (0,\ 1)\end{matrix}\right]$$

4. A series expansion

If $x_j\neq 0$ $(j=1,\ \cdots,\ n)$; $\beta_h^{(r)}(b_j^{(r)}+\sigma_r)\neq\beta_j^{(r)}(b_j^{(r)}+\nu_r)$,

for $j\neq h\ ;j,\ h=1,\ \cdots,\ M_r\ ;\sigma_r,\ \nu_r=0,\ 1,\ \cdots;\ (r=1,\ \cdots,\ n)$ *and*

$\lambda_r=\sum_{j=1}^{c}\gamma_j^{(r)}+\sum_{j=1}^{P_r}\alpha_j^{(r)}-\sum_{j=1}^{D}\delta_j^{(r)}-\sum_{j=1}^{Q_r}\beta_j^{(r)}<0,$ *for* $r=1,\ \cdots,\ n$; *then*

$$(4.1)\quad H\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \sum_{h_1=1}^{M_1} \cdots \sum_{h_n=1}^{M_n} \sum_{\nu_1=0}^{\infty} \cdots \sum_{\nu_n=0}^{\infty} x_1^*(G) \times \left\{ \prod_{r=1}^{n} x_2^*(G) \frac{(-1)^{\nu_r} z^G}{(\nu_r)!\,\beta_{h_r}^{(r)}} \right\}_{j \neq h_r}$$

where $G=\dfrac{b_{h_r}^{(r)}+\nu_r}{\beta_{h_r}^{(r)}}$ *and* x_1^* *and* x_2^* *are defined in* (1.2) *and* (1.3) *respectively.*

The result can be proved on computing the residues at the poles

$$s_r=\frac{b_{h_r}^{(r)}+\nu_r}{\beta_{h_r}^{(r)}}(h_r=1, \cdots, M_r\,;\, \nu_r=0, 1, \cdots),\ \textit{for}\ r=1, \cdots, n.$$

5. Special cases

A large number of special cases of the result(4.1) associated with products of several special functions occuring in Mathematical Physics and Chemistry, Statistics and Biological Sciences can be derived on account of the most general character of the H-function of n variables occurring there, but for the sake of brevity they are not presented here. However, a few interesting special cases are enumerated below.

(i) For $n=2$, (4.1) reduces to a result given recently by Goyal [5].

(ii) When $M_1=\cdots=M_n=1$, $C=A$, $\beta_{h_r}^{(r)}=1$, $b_{h_r}^{(r)}=0$ for all r, $Q_1=N_1$ replace $1-a_j^{(r)}$ by $a_j^{(r)}$ and $1-b_k^{(r)}$ by $b_j^{(r)}$, (4.1) then gives rise to the generalized Lauricella functions of Srivastava and Daoust [16].

(iii) Finally if we take $n=1$ then (4.1) reduces to Braaksma's formula for Fox's H-function [1].

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