On Estimating the Variance of a Normal Distribution with Known Coefficient of Variation

S. K. Ray and A. Sahai*

ABSTRACT

This note deals with the estimation of the variance of a normal distribution $N(\theta, c\theta^2)$ where c, the square of coefficient of variation is assumed to be known. This amounts to the estimation of θ^2. The minimum variance estimator among all unbiased estimators linear in $\bar{x}^2$ and s^2 where $\bar{x}$ and s^2 are the sample mean and variance, respectively, and the minimum risk estimator in the class of all estimators linear in $\bar{x}^2$ and s^2 are obtained. It is shown that the suggested estimators are BAN.

1. INTRODUCTION

Consider a normal distribution $N(\theta, c\theta^2)$ with mean θ and variance $c\theta^2$ where c, the square of the coefficient of variation, is assumed to be known. The problem is to estimate the varince or, more specifically, θ^2 on the basis of a fixed sample of size n. In this case Searles (1964), Khan (1968), and Glesser and Healey (1976) gave estimators of the mean θ indicating practical situations wherein the coefficient of variation may be known. Das (1975) considered the estimation of θ^2 and suggested;

$$T_u = \bar{x}^2 / (1 + c/n)$$

as the unbiased estimator of θ^2.

In the present paper, we treat the problem of estimating θ^2 as a decision

* Department of Statistics, Lucknow University, Lucknow, India. 226007.

problem using the squared-error loss function

$$L(\theta^2, T)=(T-\theta)^2$$

where T is an estimator of θ^2.

We assume that a random sample $x_1, x_2, \ldots, x_n$ of fixed size $(n \geqslant 2)$ is taken from the normal distribution $N(\theta, c\theta^2)$ where $\theta > 0$ is unknown and $c > 0$ is known.

Let

$$\bar{x}=n^{-1}\sum_{i=1}^{n} x_i, \quad s^2=(n-1)^{-1}\sum_{i=1}^{n}(x_i-\bar{x})^2.$$

In section 2 we consider the class of unbiased estimators linear in $\bar{x}^2$ and s^2 and obtain the minimum risk (or variance) estimator of θ^2 in this class and show that it is BAN (best asymptotically normal). In section 3 we consider the case of all estimators linear in $\bar{x}^2$ and s^2 and obtain the minimum risk estimator of θ^2 in this class and show that it is also BAN.

2. LINEAR MINIMUM VARIANCE ESTIMATOR

It is easy to check that

$$T_1=s^2/c$$

and

$$T_2=\bar{x}^2/b$$

where $b=(1+c/n)$, are unbiased estimators of θ^2. Also

$$\mathrm{Var}(T_1)=2\theta^4/(n-1)$$

$$\mathrm{Var}(T_2)=2\theta^4(b^2-1)/b^2$$

and

$$\mathrm{Covar}(T_1, T_2)=0$$

Now we consider the class of unbiased estimators of θ^2 as

$$T=kT_1+(1-k)T_2, \quad 0 \leq k \leq 1 \qquad (2.1)$$

for which the risk under the squared error loss function is given by

$$R(T,\theta^2)=E(T-\theta^2)^2$$

$$=\mathrm{Var}(T)$$

$$=k^2\text{Var}(T_1)+(1-k)^2\ \text{Var}(T_2)$$

The best value of k which minimises $R(T,\ \theta^2)$ is

$$k^*=\text{Var}(T_2)/[\text{Var}(T_1)+\text{Var}(T_2)]$$

$$=(n-1)(b^2-1)/[nb^2-(n-1)] \tag{2.2}$$

Therefore,

$$T_{LU}=k^*T_1+(1-k^*)T_2$$

$$=k^*s^2/a+(1-k^*)\bar{x}^2/b$$

where k^* is given by (2. 2), is the minimum variance estimator among all unbiased estimators of the type given by (2. 1). Hence T_{LU} is uniformly better than T_1 and T_2 alone, so that, T_n of Das (1975) is inadmissible.

It is easy to check that as $n\to\infty$, $k^*\to 2c/(1+2c)$, $\text{Var}(T_1)\ \to 2\theta^4/n$ and Var $(T_2)\to 4c\theta^4/n$. Therefore the asymptotic variance of T_{LU} is $4c\theta^4/n(1+2c)$ which happens to be the Cramer-Rao lower bound for the variance of an unbiased estimator of θ^2. This leads us to the assertion that T_{LU} is a BAN estimator of θ^2.

3. LINEAR MINIMUM RISK ESTIMATOR

Now we consider the class of all estimators of θ^2 which are linear in $\bar{x}^2$ and s^2, but not necessarily unbiased *i.e.* of the type given by

$$T'=k_1\bar{x}^2+k_2s^2 \tag{3.1}$$

for which the risk under the squared-error loss function is

$$R(T',\theta^2)=E(T'-\theta^2)^2$$

$$=E(T')^2-2\theta^2E(T')+\theta^4$$

$$=k_1^2E(\bar{x}^4)+k_2^2E(s^4)+2k_1k_2E(\bar{x}^2)E(s^2)-2\theta^4(k_1b+k_2c)+\theta^4$$

The best values of k_1, k_2 which minimise $R(T',\theta^2)$ are

$$k_1^*=b/[(2n+1)b^2-2n]$$

and (3. 2)

$$k_2^*=(2/n)(n-1)(b+1)/[(2n+1)b^2-2n]$$

Therefore,

$$T_{LM}=k_1^*\bar{x}^2+k_2^*s^2$$

where $k_1{}^*$ and $k_2{}^*$ are given by (3. 2) is the minimum risk estimator among all estimators of type given by (3. 1). Also

$$R(T_{LM},\theta^2)=2\theta^4(b^2-1)/[(2n+1)b^2-2n]$$

It may further be checked that as $n\to\infty$, $R(T_{LM},\theta^2)=4c\theta^4/n(1+2c)+0(1/n^2)$ so that the asymptotic variance of T_{LM} is $4c\theta^4/n(1+2c)$ which is the Cramer-Rao lower bound. This leads us to the assertion that T_{LM} is also BAN estimator of θ^2.

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