Stochastic Duels with Random Detection

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ABSTRACT

This paper presents a method of incorporating detection capability of weapon systems into the "fundamental" stochastic duels of Williams and Ancker when both detection and interfiring times are continuous random variables. An example with negative exponential detection and firing times is also given.

1. INTRODUCTION

A kill is obtained by firings that can be activated only after the target is completely acquired. Generally for a weapon system to be effective, it must be equipped with efficient detection devices (radar, sonar, etc.) as well as powerful firing systems. Thus, a combat model that includes a process of detection-destruction may be more effective than the one that considers only destruction.

Dubins and Morgenthaler [4] included detection in survival models for vehicle-to-vehicle space combat and Barfoot [2] incorporated random initial surprise into Markov duels. Williams and Ancker [8] extended their fundamental duel to the case of random initial surprise. However initial detection capability is not considered.

As an extension of Williams and Ancker [8], we incorporate random initial detection as well as surprise into stochastic duels. An example with negative exponential detection and firing times is also given.

2. THE PROBLEM AND ASSUMPTIONS

Duelist A has two patterns of flow in detection-destruction: One is case I (see

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Fig. 1) where A detects B before he is detected and therefore he has more time for firing than B, and the other is case II where A has less time for firing than B since he is detected before he detects B.

It is assumed that both duelists are in an arena at time zero with unloaded weapons, unlimited ammunition supply and unlimited search efforts, and operate their detection devices simultaneously.

If A detects B first (Case I) during time interval $[x, x+dx]$, where x is called the "initial detection time" for A, he start to load his first round. However, B can load his first round only after he spent time y to detect A's firing position. y will be called "reactive detection time" for B or "firing time with impunity" for A. Then this duel, immediately after A's initial detection during time interval $(x, x+dx]$, becomes the duel with random initial surprise studied by Williams and ancker [8]. Case II can be handled similarly.

In both cases the criterion of A's winning is that he kills B before B gets him. Therefore, when the duel time is limited by T, the probability $P_A(T)$

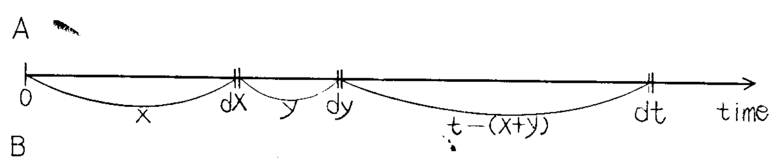

Fig. 1 The flow of detection-destruction for case I.

that A will win is the sum of two winning probabilities $P_{Ae}(T)$ of case I and $P_{Al}(T)$ of case II, that is,

(1) $P_A(T)=P_{Ae}(T)+P_{Al}(T)$.

Then, A's winning probability $P(A)$ with unlimited duel time can be defined as

$$(2)\quad P(A)=\lim_{T\to\infty}P_A(T)=P_e(A)+P_l(A)$$

where $P_e(A)$ and $P_l(A)$ are the winning probabilities for the two cases with unlimited duel time.

In this paper, general expressions for $P_A(T)$ and $P(A)$ with random detection and interfiring times are derived. In particular, two detection functions are considered as follows: One is the case of continuous detection time of Koopman [6] in which A's probability density function (pdf) of random detection time is expressed as

$$(3)\quad d_{A_1}(t) = d_A\, e^{-d_A t}$$

where d_A is A's average rate of detection. Then, the Laplace transform $d_{A_1}{}^*(s)$ of $d_{A_1}(t)$ becomes

$$(4)\quad d_{A_1}{}^*(s) = \frac{d_A}{s+d_A}$$

(Henceforth the Laplace transform of function $a(t)$ will be denoted by $a^*(s)$.) The other is the case where time between detection actions such as glimpses, sweeps or pluses is a continuous random variable. Defining $d_{A_2}(t)dt$ as the probability that A detects B in $(t, t+dt]$ and following arguments similar to Williams and Ancker [8], we obtain

$$(5)\quad d_{A_1}(t)dt = \sum_{n=i}^{\infty} \alpha_A(1-\alpha_A)^{n-1}\, \beta_A{}^{(n)}(t)dt$$

where α_A is the probability that A detects B with a detection action, $\beta_A(t)$ is A's pdf of inter-glimpse time and $\beta_A{}^{(n)}(t)$ is the n-fold convolution of $\beta_A(t)$. Then, we have

$$(6)\quad d_{A_2}{}^*(s) = \frac{\alpha_A \beta_A{}^*(s)}{1-(1-\alpha_A)\beta_A{}^*(s)}$$

3. THE MODEL

Some notations for duelist A are defined as follows;

$d_A(x)dx$: the probability of A's initial detection in $(x, x+dx]$

$g_A(y)dy$: the probability of A's reactive detection in $(y, y+dy]$

$k_A(t)dt$: the probability that A kills passive target B in $(t, t+dt]$.

$k_A'(t)$: the convolution of $g_A(t)$ and $k_A(t)$.

$k_{Ae}(t)dt$: the probability that A wins during time interval $(t, t+dt]$ when A detects B first.

$k_{Al}(t)dt$: the probabillty that A wins during time interval $(t, t+dt]$ when B detects A first.

Similar quantities for B can also be defined.

Then, from the fundamental duel of Williams and Ancker [8], we have

(7) $k_A(t)dt=\sum_{n=1}^{\infty} p_A q_A^{n-1} f_A^{(n)}(t)dt$ and $k_A^*(s)=\dfrac{p_A f_A^*(s)}{1-q_A f_A^*(s)}$

where p_A is A's single shot hit (kill) probability, $f_A(t)$ is the pdf of A's interfiring time, $f_A^{(n)}(t)$ is the u-fold convolution of $f_A(t)$ and $q_A=1-p_A$.

By definition, we also have

(8) $k_B'(t)dt=\int_0^t g_B(y).k_B(t-y)dydt=(g_B{*}k_B)\ (t)dt.$

If both detection function $d_{AB}(t)$ and kill function $k_{AB}'(t)$ are defined as

(9) $d_{AB}(t)=d_A(t)\int_t^{\infty} d_B(\tau)d\tau$ and $k_{AB}'(t)=k_A(t)\int_t^{\infty} k_B'(\tau)d\tau,$

$k_{Ae}(t)$ and $k_{Ae}^*(s)$ can be expressed as

(10) $k_{Ae}(t)dt=\int_0^t [d_A(x)\int_x^{\infty} d_B(\tau)d\tau]\cdot[k_A(t-x)\int_{t-x}^{\infty} k_B'(\tau)d\tau]dx\ dt$

$$=\int_0^t d_{AB}(x)\cdot k_{AB}'(t-x)dx\ dt$$

$$=(d_{AB}{*}k_{AB}')(t)dt,$$

and

(11) $k_{Ae}^*(s)=d_{AB}^*(s)\cdot k_{AB}'^*(s).$

Similarly, $k_{Al}(t)$ and $k_{Al}^*(s)$ are

(12) $k_{Al}(t)dt=\int_0^t [d_B(x)\int_x^{\infty} d_A(\tau)d\tau]\cdot[k_A'(t-x)\int_{t-x}^{\infty} k_B(\tau)d\tau]dx\ dt$

$$=(d_{BA}{*}k_{A'B})(t)dt,$$

and

(13) $k_{Al}^*(s)=d_{BA}^*(s)\cdot k_{AB}'^*(s)\cdot$

Then, by the definition of $P_A(T)$ in equation (1), we obtain

(14) $P_A(T)=\int_0^T (k_{Ae}(t)+k_{Al}(t))\ dt$

and

(15) $P_A^*(s)=(\frac{1}{s})[k_{Ae}^*(s)+k_{Al}^*(s)]$

$$=(\frac{1}{s})\ [d_{AB}^*(s).\ k_{AB}'^*(s)+d_{BA}^*(s).\ k_{A'B}^*(s)]$$

where $d_{AB}^*(s)$, $d_{BA}^*(s)$, $k_{AB}'^*(s)$ and $k_{AB}'^*(s)$ can be derived as follows:

By definition, $d_{AB}^*(s)$ can be written as

$$(16)\quad d_{AB}^*(s)=\int_0^\infty e^{-st}[d_A(t)\int_t^\infty d_B(\tau)d\tau]dt.$$

Changing the order of integration, it can be rewritten as

$$(17)\quad d_{AB}^*(s)=\int_0^\infty d_B(\tau)[\int_0^\tau e^{-st}d_A(t)dt]d\tau.$$

By the Mellin inversion integral [3],

$$(18)\quad \int_0^\tau e^{-st}d_A(t)dt=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{d_A^*(s+z)}{z}e^{z\tau}\,dz$$

where c is chosen in such a way as to leave all singularities of $d_A^*(s+z)$ to the left of the real line $z=c$.

From equations (17) and (18) we can write

$$(19)\quad d_{AB}^*(s)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} d_A^*(s+z)[\int_0^\infty e^{z\tau}d_B(\tau)d\tau]\frac{dz}{z}$$

$$=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} d_A^*(s+z)d_B^*(-z)\,\frac{dz}{z}.$$

Similarly, we have

$$(20)\quad d_{BA}^*(s)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} d_B^*(s+z)d_A^*(-z)\,\frac{dz}{z},$$

$$(21)\quad k_{AB}'^*(s)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} k_A^*(s+z)k_B'^*(-z)\,\frac{dz}{z}$$

$$=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} k_A^*(s+z)\ g_B^*(-z)\ k_B^*(-z)\,\frac{dz}{z},$$

and

$$(22)\quad k_{AB}'^*(s)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} g_A^*(s+z)k_A^*(s+z)\ k_B^*(-z)\,\frac{dz}{z}.$$

Evaluation of the inverse transform $P_A^*(s)$ in equation (15) is generally not an easy task. Accordingly one may use the numerical inversion techniques of Dubner and Abate [5] to compute $P_A(T)$ from $P_A^*(s)$.

From the final value theorem in Laplace transform theory, that is,

$$(23)\quad \lim_{s\to 0} sP_A^*(s)=\lim_{T\to\infty} P_A(T)$$

and $P_A^*(s)$ in equation (15), the probability $P(A)$ that A will win when the duel time is unlimited becomes

(24) $P(A)=\lim_{T\to\infty} P_A(T)$

$=D_{AB}.K_{AB}'+D_{BA}.K_{AB}'$

where

(25) $D_{AB}=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} d_A^*(z)d_B^*(-z)\ \frac{dz}{z},$

(26) $D_{BA}=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\ d_B^*(z)d_A^*(-z)\ \frac{dz}{z},$

(27) $K_{AB}'=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} k_A^*(z)g_B^*(-z)k_B^*(-z)\ \frac{dz}{z}$

(28) $K_{A}'{}_{B}=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} g_A^*(z)k_A^*(z)k_B^*(-z)\frac{dz}{z}.$

From equation (16), $D_{AB}=\int_0^\infty d_A(t)[\int_t^\infty d_B(\tau)d\tau]dt$ is the probability that A detects B before he is detected when duel time is unlimited. D_{BA}, K_{AB}', and K_{AB}' are similarly interpreted.

Equations (15) and (24) give general solutions with unspecified detection and interfiring times. As a special case, when $d_{A_1}(t)$ and $d_{A_1}^*(s)$ in equations (3) and (4) are used, $P_A^*(s)$ and $P(A)$ become

(29) $P_A^*(s)=\left(\frac{1}{s}\right)[\left(\frac{d_A}{s+d_A+d_B}\right)\cdot k_{AB}'^*(s)+\left(\frac{d_B}{s+d_A+d_B}\right)\cdot k_{AB}'^*(s)]$

(30) $P(A)=\left(\frac{d_A}{d_A+d_B}\right)\cdot K_{AB}'+\left(\frac{d_B}{d_A+d_B}\right)\cdot K_{AB}'$

If equations (5) and (6) are used and inter-glimpse times are assumed to be negative exponentials, *i.e.*, $\beta_A(t)=\beta_A e^{-\beta_A t}$ and $\beta_B(t)=\beta_B e^{-\beta_A t}$, $P_A^*(s)$ and $P(A)$ are then given by

(31) $P_A^*(s)=\left(\frac{1}{s}\right)\left[\left(\frac{\alpha_A\beta_A}{s+\alpha_A\beta_A+\alpha_B\beta_B}\right)\cdot k_{AB}'^*(s)+\left(\frac{\alpha_B\beta_B}{s+\alpha_A\beta_A+\alpha_B\beta_B}\right)\right.$

$\cdot k_{A}'{}_{B}^*(s)$

and

(32) $P(A)=\left(\frac{\alpha_A\beta_A}{\alpha_A\beta_A+\alpha_B\beta_B}\right)\cdot K_{AB}'+\left(\frac{\alpha_B\beta_B}{\alpha_A\beta_A+\alpha_B\beta_B}\right)\cdot K_{AB}'.$

Similarly for duelist B, $P_B^*(s)$ and $P(B)$ can be obtained, and the probability $P_{AB}\ (T)$ of a draw when the duel time is limited can be shown to be

$$(33)\quad P_{AB}(T)=\int_T^{\infty}\left[k_{Ae}(t)+k_{Al}(t)+k_{Be}(t)+k_{Bl}(t)\right]\,dt.$$

4. AN EXAMPLE

Let the interfiring and reactive detection times be both negative exponential;

$$f_A(t)=r_Ae^{-r_At},\qquad f_B(t)=r_Be^{-r_At},$$

$$g_A(t)=g_Ae^{-g_At},\qquad g_B(t)=g_Be^{-g_At}.$$

Then we have

$$f_A{}^*(z)=\frac{r_A}{z+r_A}\quad,\quad f_A{}^*(z)=\frac{r_B}{z+r_B},$$

$$g_A{}^*(z)=\frac{g_A}{z+g_A}\quad,\quad g_B{}^*(z)=\frac{g_B}{z+g_B}.$$

and from epuation (7),

$$k_A{}^*(z)=\frac{r_Ap_A}{z+r_Ap_A}\quad,\quad k_B{}^*(z)=\frac{r_Bp_B}{z+r_Bp_B}.$$

from these and equations (29)-(33), $P_A{}^*(s)$, $P_A(T)$, $P_{AB}(T)$ and $P(A)$ are obtained as follows:

$$(34)\quad P_A{}^*(s)=\left(\frac{1}{s}\right)\left[\left(\frac{d_A}{s+d_A+d_B}\right)\left(\frac{r_Ap_A}{r_Bp_B-g_B}\right)\left(\left(\frac{r_Bp_B}{s+r_Ap_A+g_B}\right)-\left(\frac{g_B}{s+r_Ap_A+r_Bp_B}\right)\right)+\left(\frac{d_B}{s+d_A+d_B}\right)\left(\frac{g_A}{s+r_Bp_B+g_A}\right)\left(\frac{r_Ap_A}{s+r_Ap_A+r_Bp_B}\right)\right].$$

From Heaviside expansion theorem [3], we can write

$$L^{-1}\left\{\left(\frac{1}{s}\right)\sum_{i=1}^{n}\left(\frac{1}{s-r_i}\right)\right\}=\sum_{i=1}^{n}\prod_{j\neq i}^{n}\left\{\frac{e^{r_it}}{r_i(r_i-r_j)}\right\}+(-1)^n\sum_{i=1}^{n}\left(\frac{1}{r_i}\right)$$

where L indicates a Laplace transform.

Equation (34) is then inverted to yield

$$(35)\quad P_A(T)=C_0+C_1e^{-(d_A+d_B)T}+C_2\,e^{-(r_Ap_A+r_Bp_B)T}+C_3\,e^{-(r_Ap_A+g_B)T}+C_4\,e^{-(r_Bp_B+g_A)T}$$

where

$$C_0=\left(\frac{d_A}{d_A+d_B}\right)\left(\frac{r_Ap_A}{r_Ap_A+r_Bp_B}\right)\left(\frac{r_Ap_A+r_Bp_B+g_B}{r_Ap_A+g_B}\right)+\left(\frac{d_B}{d_A+d_B}\right)$$

$$\times\left(\frac{r_A p_A}{r_A p_A + r_B p_B}\right)\left(\frac{g_A}{r_B P_B + g_A}\right),$$

$$C_1 = \frac{d_A r_A p_A (d_A + d_B - r_A p_A - r_B p_B - g_B)}{(d_A + d_B)(d_A + d_B - r_A p_A - g_B)(d_A + d_B - r_A p_A - r_B p_B)} +$$

$$\frac{d_B r_A p_A g_A}{(d_A + d_B)(d_A + d_B - r_B p_B - g_A)(r_A p_A + r_B p_B - d_A - d_B)},$$

$$C_2 = \frac{r_A p_A}{(r_A p_A + r_B p_B)(r_A p_A + r_B p_B - d_A - d_B)}\left[\frac{d_A g_B}{g_B - r_B p_B} + \frac{d_B g_A}{g_A - r_A p_A,}\right],$$

$$C_3 = \frac{d_A r_A p_A r_B p_B}{(g_B - r_B p_B)(d_A + d_B - r_A p_A - g_B)(r_A p_A + g_B)} \quad \text{and}$$

$$C_4 = \frac{d_B r_A p_A g_A}{(g_A - r_A p_A)(d_A + d_B - r_B p_B - g_A)(r_B p_B + g_A)}$$

Obviously, A's winning probability $P(A)$ with unlimited duel time is

(36) $P(A) = \lim_{T\to\infty} P_A(T) = C_0$

which can also be obtained from equations (24)-(28). We note that the first part of C_0 in equation (35) is $P_e(A)$ and the second part is $P_l(A)$.

The probability $P_{AB}(T)$ of a draw can be shown to be

(37) $P_{AB}(T) = C_1' \; e^{-(d_A+d_B)T} + C_2' \; e^{-(r_A p_A + r_B p_B)T} + C_3' \; e^{-r_A p_A + g_B)T} + C_4' \; e^{(r_B p_B + g_A)T}$

where

$$C_1' = \frac{r_A p_A (r_B p_B + g_A - d_A)}{(d_A + d_B - r_A p_A - r_B p_B)(d_A + d_B - r_B p_B - g_A)}$$

$$+ \frac{r_B p_B (r_A p_A + g_B - d_B)}{(d_A + d_A - r_A p - r_B p_B)(d_A + d_B - r_A p_A - g_B)},$$

$$C_2' = \frac{1}{(d_A + d_B - r_A p_A - r_B p_B)}\left[\frac{d_B g_A}{g_A - r_A p_A} + \frac{d_A g_B}{g_B - r_B p_B}\right],$$

$$C_3' = \frac{d_A r_B p_B}{(g_B - r_B p_B)(r_A p_A + g_B - d_A - d_B)} \quad \text{and}$$

$$C_4' = \frac{d_B r_A p_A}{(g_A - r_A p_A)(r_B p_B + g_A - d_A - d_B)}.$$

For duelist B, $P_T(B)$ and $P(B)$ are obtained by simply replacing subscripts A with B in equations (34)-(36).

For the case of negative exponential inter-glimpe times, $P_A(T)$, $P_{AB}(T)$ and $P(A)$ are evaluated by substituting $\alpha_A\beta_A$ for d_A and $\alpha_A\beta_B$ for d_B in equations (34)-(37).

Following observations are made on $P(A)$ of equation (36) as follows:

(i) If $g_A=g_B=0$, we have $P(A)=\frac{d_A}{d_A+d_B}$. That is, initial detection ability conclusively determines the winner since the detected has no chance of finding his opponent. Conversely, if $g_A=g_B=\infty$, $P(A)=\frac{r_A p_A}{r_A p_A+r_B p_B}=P_f(A)$, where $P_f(A)$ is A's winning probability of the "fundamental" duel. This means that the duelist who detects his opponent first has no firing time advantage since the opponent returns fire immediately and simultaneously.

(ii) If $d_A=d_B=0$, we have $P(A)=P(B)=0$. This means that a conbatant can not defeat his opponent because his target is not acquired at all. On the other hand if $d_A=\infty$, $d_B=0$, we have $P(A)=1$ when $g_B=0$ and $P(A)=P_f(A)$ when $g_B=\infty$.

(iii) The probabilities $P(A)$, $P_e(A)$ and $P_l(A)$ in equation (36) are plotted in Fig. 2. It is seen that these quantities vary according to initial and reactive detection rates. Here kill rates are assumed to be fixed and equal to unity $(r_A p_A=r_B p_B=1)$ to focus our attention to the detection capabilities.

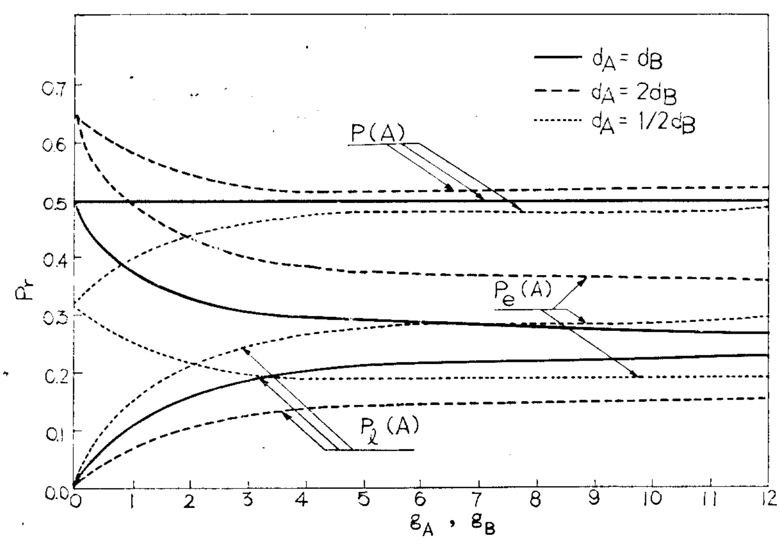

Fig. 2. Kill effects on detection

a) As g_A, $g_B \to \infty$, $P_e(A)$, $P_l(A) \to \frac{1}{4}$, $P(A) \to \frac{1}{2}$ when $d_A = d_B$,

$$P_e(A) \to \frac{1}{3}, P_l(A) \to \frac{1}{6}, P(A) \to \frac{1}{2} \text{ when } d_A = 2d_B,$$

$$P_e(A) \to \frac{1}{6}, P_l(A) \to \frac{1}{3}, P(A) \to \frac{1}{2} \text{ when } d_A = \frac{1}{2}d_B.$$

b) The winning probabilities are sensitive to detection rates less than 5 since initial detection has no time advantage for firing if reactive detection time becomes sufficiently small.

c) $P_e(A)$ shifts upward and $P_l(A)$ shifts downward as A's initial detection rate increases.

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