AN INFINITESIMAL DEFORMATION CARRYING A
HOLOMORPHICALLY PLANAR CURVE INTO A CURVE OF
THE SAME KIND IN A KAHLERIAN MANIFOLD

BY SANG-SEUP EUM

§ 1. Introduction.

In a Riemannian manifold $M$ with local coordinates $\{x^i\}$, we consider the point transformation

$$\bar{x}^i = x^i + \varepsilon v^i,$$

where $\varepsilon$ is an infinitesimal constant and $v^i$ is a vector field of $M$.

If the infinitesimal point transformation (1.1) under the condition

$$\sum g_{kj} \frac{d\varepsilon}{ds} v^j = 0,$$

where $g_{kj}$ is the Riemannian metric and $s$ is the arc-length of the curve, maps any geodesic into a geodesic, the equation of Jacobi:

$$\frac{\delta^2 v^k}{ds^2} + \mathcal{R}^k_{lij} \frac{dx^l}{ds} \frac{dx^j}{ds} = 0$$

is satisfied, where $\frac{\delta}{ds}$ denotes covariant differentiation along the curve, $\mathcal{R}^k_{lij}$ is the curvature tensor of $M$ and the terms of order higher than one with respect to $\varepsilon$ are neglected. If the solution of the equation (1.3) vanishes at a point $p_0$ and at another point $p_1$ and if it does not vanish between $p_0$ and $p_1$ then the points $p_0$ and $p_1$ are said to be conjugate on this geodesic.

Recently K. Yano and I. Mogi studied the distance between consecutive conjugate points on a geodesic in a Kaehlerian manifold and proved the following [2]

**THEOREM A.** In a Kaehlerian manifold of positive constant holomorphic curvature $k$ ($>0$), the distance between two consecutive conjugate points is constant and is given by $2\pi / \sqrt{k}$.

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On the other hand, a curve \( x^i(t) \) in a Kaehlerian manifold defined by
\[
\frac{\partial^2 x^k}{dt^2} = \alpha \frac{dx^k}{dt} + \beta \phi_j^k \frac{dx^j}{dt}
\]
is, by definition \([1]\), a holomorphically planar curve or an \( h \)-plane curve, where \( \phi_j^k \) is the Kaehlerian structure and \( \alpha, \beta \) are certain functions of \( t \).

The purpose of the present paper is to study an infinitesimal deformation carrying an \( h \)-plane curve into a curve of the same kind in a Kaehlerian manifold and to obtain a result analogous to the theorem A on an \( h \)-plane curve in a Kaehlerian manifold.

§ 2. An infinitesimal deformation carrying an \( h \)-plane curve into a curve of the same kind in a Kaehlerian manifold.

Let us consider a 2n-dimensional Kaehlerian manifold with local coordinates \( \{x^i\} \). Then the Riemannian metric \( g_{ji} \) and the Kaehlerian structure \( \phi_j^i \) satisfy the following equations
\[
\phi_k^j \phi_j^i = -\delta_k^i, \quad g_{k\ell} \phi_j^k \phi_j^\ell = g_{ji}, \quad \forall \phi_j^i = 0.
\]

In a Kaehlerian manifold, we consider a curve \( L: x^k = x^k(s) \) parameterized with its arc-length \( s \) and satisfies the differential equation
\[
\frac{\partial^2 x^k}{ds^2} = \alpha \frac{dx^k}{ds} + \beta \phi_j^k \frac{dx^j}{ds}, \quad (\alpha > 0)
\]
where \( \frac{\partial}{ds} \) indicates covariant differentiation along \( L \) and \( \alpha \) is a constant.

If we use an arbitrary parameter \( t \) of \( L \), then the equation (2.1) turns into
\[
\frac{\partial^2 x^k}{dt^2} = \alpha \frac{dx^k}{dt} + \beta \phi_j^k \frac{dx^j}{dt},
\]
where \( \alpha = -\frac{d^2 t}{ds^2}, \quad \beta = \frac{dt}{ds} \).

Since the integral curve of (2.2) is called a holomorphically planar curve \([1]\), we shall call the integral curve of (2.1) also a holomorphically planar curve or an \( h \)-plane curve in a Kaehlerian manifold.

Let \( \nu^i \) be a vector field defined along \( h \)-plane curves and assume that for any infinitesimal constant \( \varepsilon \), the point transformation:
\[
\bar{x}^i = x^i + \varepsilon \nu^i, \quad g_{kj} \nu^k \frac{dx^j}{ds} = 0
\]
maps any \( h \)-plane curve into an \( h \)-plane curve. Then we say that \( \nu^i \) preser-
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ves the h-plane curve.

Now we ask for the condition that \( v^i \) preserve the h-plane curve.

By straightforward computations, we have

\[
\frac{d^2 x^h}{ds^2} + \{k^h_j\}(\bar{x}) \frac{d\bar{x}^k}{ds} \frac{d\bar{x}^j}{ds} - a\varphi^h_j(\bar{x}) \frac{d\bar{x}^j}{ds}
\]

\[= (\bar{\partial}^h - \varepsilon v^j \{h^i\}) \left( \frac{d^2 x^i}{ds^2} - a\varphi^j_i \frac{dx^j}{ds} \right) + \varepsilon \left[ -a\varphi^h_k \frac{\partial v^k}{ds} + \frac{\partial_2 v^h}{ds^2} + K_{ik}^j \frac{dx^k}{ds} \frac{dx^j}{ds} \right].
\]

where \( \{h^i\} \) is the Christoffel symbol, \( K_{ik}^j \) is the curvature tensor of the Kaehlerian manifold and terms of order higher than one with respect to \( \varepsilon \) are neglected.

In the sequel, we always neglect terms of order higher than one with respect to \( \varepsilon \).

On the other hand, we get

\[
\left( \frac{d\bar{s}}{ds} \right)^2 = g_{kj}(\bar{x}) \frac{d\bar{x}^k}{ds} \frac{d\bar{x}^j}{ds} = 1 + 2\varepsilon \rho,
\]

where we have put

\[
(2.6) \quad \rho = g_{kj} \frac{dx^k}{ds} \frac{\partial v^j}{ds}.
\]

Using the relation (2.5), the left member of (2.4) turns into

\[
\left( \frac{d\bar{s}}{ds} \right)^2 = g_{kj}(\bar{x}) \frac{d\bar{x}^k}{ds} \frac{d\bar{x}^j}{ds} = 1 + 2\varepsilon \rho,
\]

Therefore if \( v^i \) preserves the h-plane curve then we have

\[
\frac{d\rho}{ds} \frac{dx^h}{ds} + a\varphi^h_j \frac{dx^j}{ds} = \frac{\partial_2 v^h}{ds^2} + K_{ik}^j \frac{dx^k}{ds} \frac{dx^j}{ds} - a\varphi^h_j \frac{\partial v^j}{ds^2}.
\]

From the relation (2.6), we have a system of differential equations along an h-plane curve

\[
\rho = a\varphi_{kj} \frac{dx^j}{ds},
\]

\[
\frac{d\rho}{ds} = a\varphi_{kj} \frac{\partial v^k}{ds} \frac{dx^j}{ds},
\]

\[
\frac{d^2 \rho}{ds^2} = a \left( \varphi_{kj} \frac{\partial_2 v^k}{ds^2} \frac{dx^j}{ds} + a\rho \right).
\]
§ 3. Distance between consecutive conjugate points on an $h$-plane curve in a Kaehlerian manifold of constant holomorphic curvature.

In this section, we are going to consider an infinitesimal deformation carrying an $h$-plane curve into a curve of the same kind in a Kaehlerian manifold of positive constant holomorphic curvature $k$.

In this case, the curvature tensor $K_{ik}^h$ is of the form:

(3.1) \[ K_{ik}^h = \frac{k}{4} \left( g_{kj} \delta_i^h - g_{lj} \delta_j^h + \varphi_{kj} \varphi_l^h - \varphi_{lj} \varphi_i^h - 2 \varphi_{ik} \varphi_j^h \right). \]

Substituting (3.1) into (2.7), we obtain

(3.2) \[ \frac{\partial^2 \varphi^h}{\partial s^2} - a \varphi_j^h \frac{\partial \varphi^j}{\partial s} + \frac{k}{4} \varphi^h = \left[ \frac{d\rho}{ds} \delta_j^k + (a + \frac{3}{4} k) \rho \varphi_j^h \right] \xi^i, \]

where we have put $\frac{d\xi^j}{ds} = \xi^j$.

If the solution $\varphi^h$ of the equation (3.2) vanishes at a point $p_0$ and at another point $p_1$ and if it does not vanish between $p_0$ and $p_1$ then the points $p_0$ and $p_1$ are said to be conjugate on this $h$-plane curve.

Taking account of (3.2) and the third relation of (2.8), we get

(3.3) \[ \frac{d^2 \rho}{ds^2} = -cr, \quad c = a(a + \frac{3}{4} k) + \frac{k}{4}. \]

Consequently above equation gives

(3.4) \[ \rho = A \sin \sqrt{c} \ s + B \cos \sqrt{c} \ s, \]

where $A$ and $B$ are constants.

Now we assume that $\varphi^j = 0$ and consequently $\rho = 0$ when $s = 0$. Then we have

(3.5) \[ \rho = A \sin \sqrt{c} \ s \]

from (3.4).

Substituting (3.5) into (3.2), we have

(3.6) \[ \frac{\partial^2 \varphi^h}{\partial s^2} - a \varphi_j^h \frac{\partial \varphi^j}{\partial s} + \frac{k}{4} \varphi^h = A \left[ \sqrt{c} (\cos \sqrt{c} \ s) \delta_j^k + (a + \frac{3}{4} k) (\sin \sqrt{c} \ s) \varphi_j^h \right] \xi^i, \]

$A$ being a constant.

In this place, if we put
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(3.7) \[ \frac{\partial \tau^h}{\partial s} = q^h, \quad \rho^h = q^h - b \varphi_j^h \nu^j, \]
where \( b \) is a non-zero constant given by the relation

(3.8) \[ a - b + \frac{k}{4b} = 0, \]
then we easily see that

(3.9) \[ b \nu^h = \varphi_j^h (p^j - q^j). \]

Differentiating the second relation of (3.7) and substituting it into (3.6), we obtain

(3.10) \[ \frac{\partial \rho^h}{\partial s} + \frac{k}{4b} \varphi_j^h \rho^j = A \left[ \sqrt{c} (\cos \sqrt{c} s) \partial_j \rho^h + (a + \frac{3}{4} k) (\sin \sqrt{c} s) \varphi_j^h \xi^j \right], \]
by virtue of (3.8).

Regarding (3.10) as a system of simultaneous ordinary differential equations with respect to \( \rho^h \), there exists a system of solutions \( \rho^h(x(s)) \) along an h-plane curve, and moreover this system of solutions is determined uniquely by the system of initial values \( \rho^h(x(0)) \) at the point \( s=0 \) on an h-plane curve.

On the other hand, we can see that

(3.11) \[ \rho^h = -A \left[ \frac{k}{4b} (\sin \sqrt{c} s) \xi^h + \sqrt{c} (\cos \sqrt{c} s) \varphi_j^h \xi^j \right] \]
satisfies the differential equation (3.10) along an h-plane curve.

Therefore under the system of initial conditions

(3.12) \[ \rho^h(x(0)) = -\frac{A}{a} \sqrt{c} (\varphi_j^h \xi^j)(x(0)), \]
\( \rho^h \) defined by (3.11) is a system of unique solutions of (3.11).

Substituting (3.11) into (3.9) and integrating, we can see that the system of solutions \( \tau^h \) of the system of differential equations (3.6) is determined uniquely by

(3.13) \[ \tau^h = -\frac{A}{a} (\sin \sqrt{c} s) \varphi_j^h \xi^j \]
under the system of initial conditions

(3.14) \[ \tau^h(x(0)) = 0, \quad \frac{d\tau^h}{ds}(x(0)) = -\frac{A}{a} \sqrt{c} (\varphi_j^h \xi^j)(x(0)). \]

From the first equation of (2.8) and (3.5), we can see that, if \( \tau^h \) vani-
shes at a point, then $\rho$ vanishes at this point and consequently $\sin \sqrt{-c} s$ vanishes also at this point.

Conversely, from (3.13) we can see that if $\sin \sqrt{-c} s$ vanishes at a point, then $\nu^h$ vanishes also this point.

Thus if $\nu^h$ vanishes at a point $p_0(x^h(0))$, then point at which $\sin \sqrt{-c} s$ vanishes immediately after $s=0$ is given by $s=\pi/\sqrt{-c}$. Thus we have the following

**Theorem.** In a Kaehlerian manifold of positive constant holomorphic curvature $k$, the distance between two consecutive conjugate points on an $h$-plane curve is constant and is given by $\pi/\sqrt{-c}$, where $c=a(a+\frac{3}{4}k)+\frac{k}{4}$.

If we consider the case of $a=0$ in (2.1), then the $h$-plane curve becomes a geodesic and $\sqrt{-c}$ takes the value $\sqrt{k}/2$. Therefore above theorem assures the theorem A stated in §1.

**References**


Sung Kyun Kwan University