THE LATTICE DISTRIBUTIONS INDUCED BY THE SUM OF
I.I.D. UNIFORM (0, 1) RANDOM VARIABLES

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1. Summary.
Let $X_1, X_2, \ldots, X_n$ be i.i.d. uniform (0, 1) random variables. Let $f_n(x)$ denote the probability density function (p.d.f.) of $T_n = \sum_{i=1}^{n} X_i$. Consider a set $S(x; \delta) = \{x \in \mathbb{R} : x = \delta + j, j = 0, 1, \ldots, n-1, 0 \leq \delta \leq 1\}$. The lattice distribution induced by the p.d.f. of $T_n$ is defined as follow:

$$f_n^{(\delta)}(x) = \begin{cases} f_n(x) & \text{if } x \in S(x; \delta) \\ 0 & \text{otherwise.} \end{cases}$$

In this paper we show that $f_n^{(\delta)}(x)$ is a probability function thus we obtain a family of lattice distributions $\{f_n^{(\delta)}(x) : 0 \leq \delta \leq 1\}$, that the mean and variance of the lattice distributions are independent of $\delta$.

2. Main Results:
Let $f_n(x)$ be the p.d.f. of $T_n$, then $f_n(x)$ can be written, See Wilks [1962].

$$f_n(x) = \frac{1}{(n-1)!} \sum_{i=0}^{n-1} (-1)^i \binom{n}{i} (x-i)^{n-1},$$

where

$$x_+ = \begin{cases} x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

First we show that $f_n^{(\delta)}(x)$ defined by (1) is probability function.

**THEOREM 1.** Let $f_n^{(\delta)}(x)$ be a function defined in (1). Then

$$\sum_{x \in S(x; \delta)} f_n^{(\delta)}(x) = 1.$$ 

**Proof:** Using (2), we can write

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By rearranging the summation it can be shown that

\[ \sum_{j=0}^{n-1} (-1)^{j} \binom{n}{j} (j-i)^{n-1-k} = 0 \quad \text{if} \quad 0 < k \leq n-1 \]
\[ \sum_{j=0}^{n-1} (-1)^{j} \binom{n}{j} (n-1-j)^{n-1-k} = (n-1)! \quad \text{if} \quad k = 0 \]

Hence the conclusion of theorem 1 follows.

Note that the expression (3) is a polynomial in \( \delta \) of degree \( (n-1) \) and the coefficients of \( \partial^{k} \), for \( k \geq 1 \), vanish.

To obtain the moments of probability function \( f_{n}^{(3)}(x) \), we need the following lemma.

**LEMMA:** For any positive integer \( m \) and \( r \), we have

\[ \sum_{i=0}^{m} \sum_{j=0}^{r} (-1)^{j} \binom{m+1}{i} (\delta+j-i)^{r} = \sum_{i=0}^{r} \binom{r}{q} \partial^{q} \sum_{i=0}^{m} (-1)^{j} \binom{m}{l} (m-l)^{r-q} \]
\[ = \sum_{i=0}^{r} \binom{r}{q} \partial^{q} \sum_{i=0}^{m} S(m, i) \]

where \( S(t, m) \) is Stirling number of the second kind defined by

\[ n! S(r, n) = \sum_{j=0}^{n} (-1)^{j} \binom{n}{j} (n-j)^{r} \]
\[ t^{n} = \sum_{r=0}^{n} t^{(r)} S(r, n), \quad \text{where} \quad t^{(r)} = t(t-1) \cdots (t-r+1). \]

Now we evaluate the \( k \)-th moment of the probability function \( f_{n}^{(3)}(x) \)

\[ \mu_{k} = (1/(n-1)!) \sum_{j=0}^{n-1} \sum_{i=0}^{k-1} (-1)^{j} \binom{n}{i} (\delta+j-i)^{n-1}(\delta+j)^{k} \]
\[ = \frac{1}{(n-1)!} \sum_{j=0}^{n-1} \sum_{i=0}^{k-1} (-1)^{j} \binom{n}{i} (\delta+j-i)^{n-1} \sum_{l=0}^{k} \binom{k}{l} (\delta+j-i)^{k-l} i^{l} \]
\[ = \frac{1}{(n-1)!} \sum_{l=0}^{k} \binom{k}{l} \sum_{r=0}^{l} S(l, r) \sum_{j=0}^{r-1} \sum_{i=0}^{l-1} (-1)^{j} \binom{n}{i} (\delta+j-i)^{n+k-l-1} i^{r} \]
The lattice distributions induced by the sum of i.i.d. uniform (0,1) random variables

\[
\sum_{q=0}^{n+k-1-q} \sum_{j=0}^{n-j-q} (-1)^j \binom{n-r-1}{j} (n-r-j)^{n+k-1-q-j} \times
\]

\[
\sum_{r=0}^{l} S(l, r) (-1)^r n^r \sum_{i=0}^{n+k-1-q} \binom{k}{i} \frac{(n+k-1-l)}{q} \times
\]

\[
\sum_{r=0}^{l} S(l, r) (-1)^r n^r \sum_{i=0}^{n+k-1-q} \binom{k}{i} \frac{(n+k-1-l)}{q} \times
\]

\[
\sum_{r=0}^{l} S(l, r) (-1)^r n^r \sum_{i=0}^{n+k-1-q} (n-r-1)! S(t, n-r-1)
\]

Using (6) in conjunction with the properties of Stirling number of the

second kind, the following theorem can be established.

**Theorem 2:** The mean and variance of the p.f. \( f_n^{(\delta)}(x) \) is independent of \( \delta \) if \( n \geq k+1 \) for \( k=1, 2 \). That is,

\[\mu = \mu_1' = n/2, \quad \mu_2' = n(3n+1)/12, \quad \text{and} \]

\[\sigma^2 = \mu_2' - \mu^2 = n/12.\]

We note that the mean and variance of \( f_n^{(\delta)}(x) \), \( \delta \neq 0 \), is same as the

mean and variance of \( f_n^{(0)}(x) \), for \( n \geq 3 \).

However we have not obtained \( \mu_2' \) for \( k \geq 3 \) and can not conclude whether

or not they are also independent of \( \delta \). It would be interesting to find the

set of values of \( k \) such that the \( k \)-th moment of \( f_n^{(\delta)}(x) \) is independent of \( \delta \).

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