HOMGREN'S THEOREM AND ITS APPLICATION
TO THE WAVE OPERATOR

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Introduction.

In this paper we shall generalize the classical Homgren's theorem on the uniqueness of $C^m$-function solution of the Cauchy problem of the $m$-th order with analytic coefficients. Our generalization (cf. § 2) results in the uniqueness of the solution, $C^m$-function of $t$ valued in $\mathcal{D}'(Q_x)$, of the Cauchy problem of the $m$-th order with analytic coefficients. This result mainly based on the recent results of Treves [6] on the abstract version of the Homgren's theorem.

We shall also show that the above generalization can be usefully applied in the wave operator theory (cf. § 3) in deriving the supports of the fundamental solutions of the wave operator and the uniqueness of the solutions, $C^2$ function of $t$ valued in $\mathcal{D}'(R_x)$, of the wave equation with the initial data in $\mathcal{D}'(R_x)$.

1. Notations and definitions.

Let $\mathbb{R}^n$ be the $n$-dimensional Euclidean space. The element in $\mathbb{R}^n$ will be denoted by $x = (x_1, \ldots, x_n)$. We denote by $\mathbb{R}^n$, the dual of $\mathbb{R}^n$ and its element will be denoted by $\xi = (\xi_1, \ldots, \xi_n)$. When we want to specialize a special component, we shall sometimes use the notation $(x, t)$ to be an element in $\mathbb{R}^n \times \mathbb{R}$ as $(x^1, \ldots, x^n, t)$ and $(\xi, \tau)$ to be an element in the dual space $\mathbb{R}^n \times \mathbb{R}$ as $(\xi^1, \ldots, \xi^n, \tau)$.

A linear partial differential operator in $n$-independent variables $x = (x^1, \ldots, x^n)$ with complex coefficients defined in an open subset $\mathcal{O}$ of $\mathbb{R}^n$ is a polynomial in the partial differentiations and has the form

$$P(x, D_x) = \sum_{|\alpha| \leq m} c_\alpha(x) D_x^\alpha.$$ 

Here $\alpha$ is a multi-index, that is, an $n$-tuples of integers $\alpha_j \geq 0$; $|\alpha|$ denotes its length $\alpha_1 + \cdots + \alpha_n$. Also

$$D_x^\alpha = (D_1)^{\alpha_1} \cdots (D_n)^{\alpha_n}.$$
where $D_t^a := \left(-i \frac{\partial}{\partial x^1}\right)^a i$.

When $P(x, D_x)$ is a partial differential operator defined in $\Omega$, then the function $P_m(x, \xi)$ on $\Omega \times \mathbb{R}^n$ is called the principal symbol where $P_m(x, \xi)$ is such that

$$P_m(x, D_x) = \sum_{|\alpha| = m} c_\alpha(x) D_x^\alpha.$$

Fixing $x$ in $\Omega$, we consider the set of zeros of $P_m(x, \xi)$ as a function of $\xi$:

$$C_P(x) = \{ \xi \in \mathbb{R}^n \mid P_m(x, \xi) = 0 \}.$$

**Definition 1.1.** The set $C_P(x)$ is called the characteristic cone of $P(x, D_x)$ at the point $x \in \Omega$. Every covector $\xi \in C_P(x)$ different from zero is said to be characteristic with respect to $P(x, D_x)$ at the point $x$.

Let $S$ be a $C^1$-hypersurface in $\Omega$. By this we mean a subset $S$ of $\Omega$ such that for every $x_0$ of $\Omega$ has an open neighborhood $U_0$ where there is a $C^1$-function $\varphi(x)$ in $U_0$ with the following property: gradient of $\varphi$ does not vanish anywhere in $U_0$ and $S \cap U_0$ is exactly the set of points $x \in U_0$ such that $\varphi(x) = 0$.

**Definition 1.2.** The hypersurface $S$ is said to be noncharacteristic if any normal covector $\xi$ to $S$ at any point $x$ of $S$ is noncharacteristic.

We shall denote by $\Delta_x$ the Laplacian operator

$$\left(\frac{\partial}{\partial x^1}\right)^2 + \ldots + \left(\frac{\partial}{\partial x^n}\right)^2.$$

$\mathcal{D}'(\mathbb{Q})$ will be the space of all distributions in the open subset $\mathbb{Q}$ of $\mathbb{R}^n$. For details of unexplained definitions and notations we refer to Treves [7].

**2 General version of the classical Høngren’s theorem.**

In this section we shall generalize the classical Høngren’s theorem. Let $\Omega$ be an open subset of $\mathbb{R}^n$ and $(-T, T)$ be an open interval in $\mathbb{R}^1$. Let a partial differential operator $P(y, D_y)$ ($y = (x, t)$) is of the type

$$D_t^m + \sum_{\sigma_0 + |\sigma| \leq m} c_{\sigma_0, \sigma}(x, t) D_t^{\sigma_0} D_x^\sigma,$$

where $y = (x, t) \in \Omega \times (-T, T) \subset \mathbb{R}^{n+1}$.

We assume that the coefficient $c_{\sigma_0, \sigma}(x, t)$ are analytic in $\Omega \times (-T, T)$.

Then the classical Høngren’s theorem states that

if $u$ is a $C^m$-function of $(x, t)$ satisfying (2.1) in $\Omega \times (-T, T)$ and
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\[ D_t^k u = 0 \text{ for all } x \in \Omega \text{ and } k = 0, 1, 2, \ldots m - 1, \text{ then } u = 0 \text{ in the neighborhood of } \Omega \times \{0\}. \]

The classical Homgren's theorem has been investigated in many various ways. An outstanding one of them is the abstract version of the Homgren's theorem derived from the study of the uniqueness results of linear partial differential equations which follows from the dual form of the abstract Cauchy-Kovalevska theorem (cf. [6]). The abstract version of the Homgren's theorem applied in the case of a single partial differential operator of order \( m > 0 \), \( P(y, D_y) \) is the variable in an open subset \( U \) of \( \mathbb{R}^N \) can be stated as follows.

**Theorem 2.1.** Suppose that the coefficient of the differential operator \( P(y, D_y) \) are analytic in the open set \( U \). Let \( \Sigma \) be a \( C^1 \)-hypersurface in \( U \) subdividing \( U \) into two parts and nowhere characteristic with respect to \( P(y, D_y) \).

Then there is an open neighborhood of \( \Sigma \) in \( U \), \( \mathcal{U} \), such that every distribution \( u \) in \( U \), satisfying there \( P(y, D_y) u = 0 \) and vanishing on one side of \( \Sigma \), also vanishes in \( \mathcal{U} \).

In the case when the partial differential operator \( P(y, D_y) \) is of type (2.1), we can use the theorem 2.1 to derive a general version of the classical Homgren's theorem applied to the solution of (2.1) which is a \( C^m \)-function of \( t \) valued in \( \mathcal{D}'(\Omega) \).

**Theorem 2.2.** Let \( u \) be a \( C^m \)-function of \( t \) valued in \( \mathcal{D}'(\Omega_x) \) satisfying \( P(y, D_y) u = 0 \) in \( \Omega \times (-T, T) \) where \( P(y, D_y) \) is of the type (2.1) with analytic coefficients. If \( D_t^k u = 0 \) when \( t = 0 \) for all \( k = 0, 1, \ldots, m - 1 \) in \( \Omega \), then \( u = 0 \) in some neighborhood of \( \Omega \times \{0\} \).

**Proof.** Consider \( \bar{u}(x, t) = H(t) u(x, t) \) where \( H \) is the Heaviside's function, equal to one on the positive half-line and to zero on the negative half-line. Since \( u \) is a \( C^m \)-function of \( t \) and \( D_t^k u = 0 \) when \( t = 0 \) for all \( k = 0, 1, \ldots, m - 1 \), we have (cf. [5])

\[ D_t D_t^k u = H(t) D_t^k u, \quad k = 0, 1, \ldots, m, \]

and hence \( \bar{u} \) also satisfies \( P(y, D_y) u = 0 \). Of course, by the definition of \( \bar{u} \), \( \bar{u} = 0 \) for \( t < 0 \). Since the \( C^1 \) hypersurface \( t = 0 \) is nowhere characteristic with respect to \( P(y, D_y) \), we can apply the theorem 2.1 to complete our proof.

### 3. Applications to the wave operator.

In this section we shall make applications of the theorem 2.2 to the wa-
ve operator. We recall first that one of the fundamental solutions of the wave operator

\[ \frac{\partial^2 u}{\partial t^2} - \Delta x u \]

in \( R^{n+1} \) is

\[ E(x, t) = H(t) \mathcal{U}(x, t) \]

where \( \mathcal{U}(x, t) \) is a distribution whose Fourier transform with respect to \( x \)-variable is \( \sin(|\xi| t) / |\xi| \) and \( H(t) \) is the Heaviside function. Since \( H(t) = 0 \) when \( t < 0 \), \( E(x, t) \) has its support in the upper half space \( t \geq 0 \). Let us define the forward light cone \( \Gamma_+ \) as

\[ \Gamma_+ = \{(x, t) \in R^{n+1} | |x|^2 \geq t^2, \ t \geq 0 \}. \]

We shall show that an application of the theorem 2.1 says that any fundamental solution of the wave operator, which has its support in the upper half space \( t \geq 0 \), has its supports in \( \Gamma_+ \). Thus in particular, \( E(x, t) \) has its support in \( \Gamma_+ \).

To see this, let us recall that the fundamental solutions of the wave operator is the solution of the linear partial differential equation

\[ \frac{\partial^2 u}{\partial t^2} - \Delta x u = \delta \]

where \( \delta \) is the Dirac measure at origin, it follows that if \( \Omega \) is any open subset of \( R^n \) which does not contain origin, then in \( \Omega \times (-T, T) \) the fundamental solutions are the solutions of the homogeneous wave equation

\[ \frac{\partial^2 u}{\partial t^2} - \Delta x u = 0. \]

**Theorem 3.1.** Any fundamental solution of the wave operator, whose support lies in the upper half space \( t \geq 0 \), has its support in the forward light cone \( \Gamma_+ \).

**Proof.** It suffices to prove that the fundamental solution vanishes identically in the complement of the forward light cone \( \Gamma_+ \). The characteristic hypersurface of the wave operator is the hypersurface whose normal covector at every point of the hypersurface lies on the characteristic cone \( |\xi|^2 = \tau^2 \) where \( (\xi, \tau) \in R_{n+1} \).

Let \((x_0, t_0)\) belongs to the complement of \( \Gamma_+ \). Then we can find a non-characteristic \( C^1 \)-hypersurface \( S^1 \) passing \((x_0, t_0)\) and intersecting with the non-characteristic \( C^1 \)-hypersurface \( \tau = 0 \), which we shall call \( S^0 \), in the bound-
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dary of a compact set $K$ in $\mathbb{R}^n$ such that $K$ does not contain the origin.
(Geometrically obvious construction of $S^1$ can be done by the $C^1$-partitions of unity. cf. Fig. 1)

We can deform $S^1$ continuously to $S^0$ by noncharacteristic $C^1$-hypersurfaces $S^\alpha (0<\alpha<1)$ such that $S^\alpha \cap S^0$ equals to the boundary of $K$ and in the upper half space $t\geq 0$, $S^0$ lies above $S^\alpha$ if $\alpha<\beta$. Since the fundamental solution vanishes on the lower half space $t<0$, by the theorem 2.1, the fundamental solution vanishes in the neighborhood of $K$. As $K$ is compact, there exists $\alpha (0<\alpha \leq 1)$ such that the fundamental solution vanishes on the closure of the region $D_\alpha$ surrounded by $S^\alpha$ and $S^0$. Now if we apply the theorem 2.1 to the noncharacteristic $C^1$-hypersurface $S^\alpha$, we conclude that there exists $S^\beta (\alpha<\beta \leq 1)$ such that the fundamental solution vanishes on the closure of the region $D_\beta$ surrounded by $S^\beta$ and $S^0$. Let $\gamma (0<\gamma \leq 1)$ be the upper bound such that the fundamental solution vanishes on the closure of $D_\gamma$ surrounded by $S^\gamma$ and $S^0$. Then it follows that $\gamma = 1$, since otherwise we can apply the theorem 2.1 on $S^\gamma$ to get a contradiction. This shows that the fundamental solution vanishes at $(x_0, t_0)$, thus completing the proof.

In the remaining of this section we shall show that the theorem 2.2 can be applied to deduce that the Cauchy problem of the wave equation has the unique solution as $C^2$-function of $t$ valued in $\mathcal{D}'(\mathbb{R}^n)$. Thus we think of

\begin{align}
(3.2) & \quad \frac{\partial^2 u}{\partial t^2} - \Delta u = f(x, t) \quad \text{in } \mathbb{R}^{n+1}, \\
(3.3) & \quad \text{when } t=0, \quad u=u_0(x), \quad \frac{\partial u}{\partial t} = u_1(x) \quad \text{in } \mathbb{R}^n.
\end{align}

We shall use the following existence part of solution of (3.2)-(3.3) (cf. [7]):

Let $u_0, u_1$ be any two distributions in $\mathbb{R}^n$, $f(x, t)$ any continuous function of $t$ valued in $\mathcal{D}'(\mathbb{R}^n)$. There is a $C^2$-function of $t$ valued in $\mathcal{D}'(\mathbb{R}^n)$ which is
a solution of \((3.2)-(3.3)\).

**Theorem 3.2.** Let \(u_0, u_1\) be any two distributions in \(\mathbb{R}^n\), \(f(x, t)\) any continuous function of \(t\) valued in \(\mathcal{D}'(\mathbb{R}_x^n)\). Then the Cauchy problem \((3.2)-(3.3)\) of the wave equation has a unique solution as a \(C^2\)-function of \(t\) valued in \(\mathcal{D}'(\mathbb{R}_x^n)\).

**Proof.** Let \(v(x, t)\) and \(w(x, t)\) be any two solutions of \((3.2)-(3.3)\) as \(C^2\)-functions of \(t\) valued in \(\mathcal{D}'(\mathbb{R}_x^n)\). We note that the characteristic cone of the wave operator is \(|\xi|^2 + |\tau|^2 = 1\). Therefore if \((x_0, t_0)\) \((t_0 > 0)\) is any point in \(\mathbb{R}^{n+1}\), we can find a noncharacteristic \(C^1\)-hypersurface \(S^1\) passing \((x_0, t_0)\) and intersecting with the noncharacteristic \(C^1\)-hypersurface \(t = 0\), say \(S^0\), in the boundary of a compact set \(K\). (cf.)

![Fig. I](image)

We can deform \(S^1\) continuously to \(S^0\) by the noncharacteristic \(C^1\)-hypersurface \(S^\alpha\) \((0 < \alpha < 1)\) such that \(S^\alpha \cap S^0\) is the boundary of \(K\) and \(S^0\) lies above \(S^\alpha\), if \(\alpha < \beta\), in the upper half space \(t \geq 0\). Since \(u = v - w\) is a \(C^2\)-function of \(t\) valued in \(\mathcal{D}'(\mathbb{R}_x^n)\) and \(u = 0, \frac{\partial u}{\partial t} = 0\) when \(t = 0\), we can apply the theorem 2.2 to conclude that \(u\) vanishes on the closure of the region \(D_\alpha\) surrounded by \(S^\alpha\) and \(S^0\) for some \(\alpha\) \((0 < \alpha \leq 1)\). Now if we apply the theorem 2.1 to the hypersurface \(S^\alpha\), we see that there exists \(\beta\) \((\alpha < \beta \leq 1)\) such that \(u\) vanishes on the closure of \(D_\beta\), the region surrounded by \(S^\beta\) and \(S^0\). Let \(\gamma\) \((0 < \gamma \leq 1)\) be the upper bound such that \(u\) vanishes on the closure of \(D_\gamma\). Then \(\gamma = 1\) as is shown in the proof of the theorem 2.1. Since \((x_0, t_0)\) \((t_0 \geq 0)\) is arbitrary and the symmetric argument can be applied for \((x_0, t_0)\) \((t_0 < 0)\), this completes our proof.

**References**

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