SOME REMARKS ON $DCS/x$ SPACES

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1. Introduction.

In paper [2], N. L. Levine proved that for an invertible spaces certain local properties become global properties.

V. M. Klassen introduced $DCS$ space and $DCS/x$ space. $DCS/x$ space has the above property. We investigate some properties in $DCS/x$ spaces.

DEFINITION 1.1. [1]. A topological space $X$ is said to have the disappearing closed set (DCS) property or to be a $DCS$ space, if for every proper closed subset $C$ there is a family of open sets $\{U_i\}_{i=1}^{\infty}$ such that $U_{i+1} \subseteq U_i$ and $\bigcap_{i=1}^{\infty} U_i = \emptyset$, and there is also a sequence $\{h_i\}_{i=1}^{\infty}$ of homeomorphisms on $X$ onto $X$ such that $h_i(C) \subseteq U_i$ for all $i$.

DEFINITION 1.2. [1]. A topological space $X$ is said to have $DCS/x$ property or to be a $DCS/x$ space, if for every proper closed subset $C$ which miss $x$ there exist two sequences $\{U_i\}_{i=1}^{\infty}$ and $\{h_i\}_{i=1}^{\infty}$ satisfying the $DCS$ property.

2. Main results.

LEMMA 2.1. For every neighborhood $P$ of $x$ there is a sequence $\{h_i\}_{i=1}^{\infty}$ of homeomorphisms on $X$ onto $X$ such that $\bigcup_{i=1}^{\infty} h_i(P) = X$.

Proof. Let $\{U_i\}_{i=1}^{\infty}$ be a decreasing sequence of open sets in $X$ such that $\bigcap_{i=1}^{\infty} U_i = \emptyset$ and $\{h_i\}_{i=1}^{\infty}$ a sequence of homeomorphisms in $X$ such that $h_i(X-P) \subseteq U_i$ for each $i$. Then $X - \bigcup_{i=1}^{\infty} h_i(P) \subseteq \bigcap_{i=1}^{\infty} U_i$, so $X \subseteq \bigcup_{i=1}^{\infty} h_i(P)$ since $\bigcap_{i=1}^{\infty} U_i = \emptyset$.

THEOREM 2.2. If $P$ satisfies the first axiom of countability then $X$ satisfies the first axiom of countability.

Proof. Let $a \in X$ and $U$ be an open neighborhood of $a$. Let $\{h_i\}_{i=1}^{\infty}$ be a sequence of homeomorphisms in $X$ such that $\bigcup_{i=1}^{\infty} h_i(P) = X$. Then $a \in h_{i_0}(P)$ for some integer $i_0$ and thus $h_{i_0}^{-1}(a) \in P$. Let $\{U_j\}_{j=1}^{\infty}$ be a countable open
base of $h_0^{-1}(a)$ in $P$. Then $h_0^{-1}(a) \subseteq U_j \cap h_0^{-1}(U) \cap P$ for some integer $j$. Hence \{ $h_0(U_j) | j=1, 2, \ldots$ \} is a countable open base for $a$ in $X$.

**Theorem 2.3.** If $P$ satisfies the second axiom of countability, then $X$ satisfies the second axiom of countability.

**Proof.** Let \{ $U_i$ \}$_{i=1}^{\infty}$ be a countable base in $P$ and let \{ $h_j$ \}$_{j=1}^{\infty}$ be a sequence of homeomorphisms in $X$ such that $\bigcup_{j=1}^{\infty} h_j(P) = X$. Then \{ $h_j(U_i) | i, j=1, 2, \ldots$ \} is a countable base in $X$ since \{ $h_j(U_i) | i=1, 2, \ldots$ \} is a base in $h_j(P)$ for each $j$.

**Theorem 2.4.** If $P$ is a Lindelöf subspace of $X$ then $X$ is Lindelöf.

**Proof.** Let \{ $U_a$ \} be an open covering of $X$ and let \{ $h_i$ \}$_{i=1}^{\infty}$ be a sequence of homeomorphisms in $X$ such that $\bigcup_{i=1}^{\infty} h_i(P) = X$. Since $P$ is Lindelöf, $h_i(P)$ is Lindelöf for each $i$. Then there is a countable open subcovering \{ $U_i^j | j=1, 2, \ldots$ \} of \{ $U_a$ \} such that $h_i(P) \subseteq \bigcup_{j=1}^{\infty} U_i^j$ for each $i$. Therefore \{ $U_i^j | i, j=1, 2, \ldots$ \} is a countable open subcovering of \{ $U_a$ \} such that $\bigcup_{i,j} U_i^j = X$.

**Lemma 2.5.** If $P$ (or $\overline{P}$) is a connected subspace of $X$ then $P$ (or $\overline{P}$) is not clopen subset of $X$.

**Proof.** Suppose $P$ is closed subset of $X$. Then $h_i(P)$ is clopen subset of $X$ for each $i$, where \{ $h_i$ \}$_{i=1}^{\infty}$ is a sequence of homeomorphisms in $X$ for $X-P$. Then $X=\bigcup_{i=1}^{\infty} h_i(P)$ and $X$ is disjoint union. Hence there exists an integer $i_0$ such that $x \in h_{i_0}(P)$ and $x \not\in X \cup h_i(P)$. That is, $x \in X-h_i(P)$ for each $i \neq i_0$. Let \{ $U_i$ \}$_{i=1}^{\infty}$ be a decreasing sequence of open sets in $X$ such that $\bigcap_{i=1}^{\infty} U_i = \phi$ for $X-P$. Then $x \in h_i(X-P) \subseteq U_i$ for each $i \neq i_0$. Since \{ $U_i$ \} is decreasing sequence, $x \in \bigcap_{i=1}^{\infty} U_i$. It is contradict to $\bigcap_{i=1}^{\infty} U_i = \phi$.

From the above lemma we obtain the following theorem.

**Theorem 2.6.** If $P$ or $\overline{P}$ is connected subspace of $X$ then $X$ is connected.

**Proof.** Assume that $X$ is disconnected. Then there is a nonempty proper clopen subset $O$ of $X$. $O$ is neither $P$ nor $\overline{P}$ by Lemma 2.5. Since $X=\bigcup_{i=1}^{\infty} h_i(P)$ and $\phi \neq 0 \subseteq X$, there exists an $i_0$ such that $\phi \neq h_{i_0}(P) \cap 0 \subseteq h_{i_0}(P)$. Suppose $h_i(P) \supset h_i(P) \cap O$ for all $i$, then $h_i(P) \supset O$ for all $i$, so $X=0$. Hence
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**Theorem 2.7.** If \( \bar{P} \) is regular and some \( U_n \) is regular then \( X \) is regular, where \( \{U_n\} \) is a sequence in \( X \) for \( X-P \) by DCS/x property.

**Proof.** Let \( a \in X \) and \( C \) a closed in \( X \) such that \( a \notin C. \)

Case I: \( C \subset X-P \). (i) \( a \in X-P \). Since \( U_n \) is regular and \( h_n(a), h_n(C) \subset U_n \), there exists two disjoint neighborhoods \( U, V \) of \( h_n(a), h_n(C) \) in \( U_n \) respectively. Hence \( h_n^{-1}(U) \) and \( h_n^{-1}(V) \) are disjoint neighborhoods of \( a, C \) in \( X \) respectively. (ii) \( a \in P \). Put \( C_1 = C \cap \bar{P} \). If \( C_1 = \phi \), then \( a \in P \). \( C \subset X-P \) and \( X-P \) are disjoint opens in \( X \). If \( C_1 \neq \phi \), then there are two disjoint opens \( U', V' \) in \( P \) such that \( a \in U' \) and \( C \subset V' \). Let \( U, V \) be opens in \( X \) such that \( U' = U \cap \bar{P} \) and \( V' = V \cap \bar{P} \). Then \( a \in U \cap P \) and \( U \cap P \) \( \cup (X-P) \) are disjoint opens in \( X \).

Case II: \( C \cap P \neq \phi \). (i) \( a \in P \). This case is same to case I, (ii). (ii) \( a \in \bar{P}-P \). Let \( C_1 = C \cap P \), \( C_2 = C \cap (X-P) \). Then there are two disjoint neighborhoods \( U_1', V_1' \) of \( a, C \) in \( \bar{P} \) respectively and two disjoint neighborhoods \( U_2, V_2 \) of \( a, C \) in \( X \) respectively. Let \( U_1 \) and \( V_1 \) are opens in \( X \) such that \( U_1' = U \cap \bar{P} \) and \( V_1' = V \cap \bar{P} \). Then \( U_1 \cap U_2 \) \( \cup (V_1 \cap P) \) are disjoint opens in \( X \) such that \( a \in U_1 \cap U_2 \) and \( C \subset V_2 \cup (V_1 \cap P) \). (iii) \( a \in X-\bar{P} \). Let \( U, V \) are two disjoint neighborhoods of \( a \) and \( C \) in \( X \). Then \( U \cap (X-\bar{P}) \) and \( V \cap P \) are two disjoint open neighborhoods of \( a, C \) in \( X \).

Case III: \( C \subset P \). (i) \( a \in \bar{P} \). Let \( U', V' \) be disjoint neighborhoods of \( a, C \) in \( \bar{P} \) respectively and let \( U, V \) be two opens in \( X \) such that \( U' = U \cap \bar{P} \) and \( V' = V \cap \bar{P} \). Then \( U \cap V \) are disjoint opens in \( X \) such that \( a \in U \) and \( C \subset V \). (ii) \( a \in X-\bar{P} \). \( X-\bar{P} \) and \( P \) are disjoint neighborhoods of \( a, C \) in \( X \) respectively.

**Theorem 2.8.** If \( \bar{P} \) is normal and some \( U_n \) is normal, then \( X \) is normal, where \( \{U_n\} \) is a sequence in \( X \) for \( X-P \) by DCS/x property.

**Proof.** Let \( C_1, C_2 \) be disjoint closed subsets of \( X \).

Case I: \( C_1, C_2 \subset X-P \). Since \( h_n(C_1), h_n(C_2) \subset U_n \) and \( h_n(C_1) \cap h_n(C_2) = \phi \), we can take two disjoint neighborhoods of \( C_1, C_2 \) in \( X \).

Case II: \( C_2 \cap P \neq \phi \). Let \( F_1 = C_1 \cap \bar{P}, F_2 = C_2 \cap \bar{P}, G_1 = C_1 \cap (X-P) \) and \( G_2 = C_2 \cap (X-P) \). Let \( U_1', V_1' \) be disjoint neighborhoods of \( F_1, F_2 \) in \( \bar{P} \) and \( U_2, V_2 \) be disjoint neighborhoods of \( G_1, G_2 \) in \( X \). (i) \( C_1 \subset X-P \). (\( U_1 \cap U_2 \) \( \cup (U_2 \cap (X-\bar{P})) \)) and \( V_2 \cup (V_1 \cap P) \) are disjoint neighborhoods of \( C_1, C_2 \) in \( X \), where \( U_1, U_2 \) are opens in \( X \) such that \( U_1' = U_1 \cap \bar{P} \) and \( V_1' = V_1 \cap \bar{P} \). (ii) \( C_1 \cap P \neq \phi \). Put \( W_1 = U_1 \cap P, W_1' = V_1 \cap P, W_2 = U_1 \cap U_2, W_2' = V_1 \cap V_2 \), and...
Then \( C_1 \subset W_1 \cup W_2 \cup W_3 \), \( C_2 \subset W'_1 \cup W'_2 \cup W'_3 \), \( W_1 \cup W_2 \cup W_3 \), \( W'_1 \cup W'_2 \cup W'_3 \) are disjoint opens in \( X \). (iii) \( C_1 \subset P \). \( W_1 \cup (X-P) \) are disjoint open neighborhoods of \( C_1 \), \( C_2 \) in \( X \) respectively.

Case III; \( C_2 \subset P \). (i) \( C_1 \subset P \). It is trivial.
(ii) \( C_1 \cap P \neq \emptyset \). It is same to Case II (iii).
(iii) \( C_1 \subset X - P \). \((U_1 \cap P) \cup (X - P)\) and \( V_1 \cap P\) are two disjoint neighborhoods of \( C_1 \), \( C_2 \) in \( X \).

**Theorem 2.9.** If \( X \) and \( Y \) are topological spaces with \( DCS/x \) property and \( DCS/y \) property respectively, then \( X \times Y \) is \( DCS/(x, y) \) space.

**Proof.** Let \( C \) be a proper closed subset of \( X \times Y \) such that \( (x, y) \notin C \), and let \( x \in P \subset X \), \( y \in Q \subset Y \) be open sets in \( X \) and \( Y \), respectively, such that \( (x, y) \in P \times Q \subset (X \times Y) - C \). Let \( \{U_i\}_{i=1}^{\infty}, \{h_i\}_{i=1}^{\infty} \) and \( \{V_i\}_{i=1}^{\infty}, \{k_i\}_{i=1}^{\infty} \) be the open sets and homeomorphisms for \( X - P \) and \( Y - Q \) in \( X \) and \( Y \), respectively. We define a sequence of homeomorphisms in \( X \times Y \)

\[
\phi_i(a, b) = \{h_i(a), k_i(b)\} \text{ for each } (a, b) \in X \times Y,
\]

and

\[
\{W_i\}_{i=1}^{\infty} = \{(U_i \times Y) \cup (X \times V_i)\}_{i=1}^{\infty}.
\]

Then \( \{W_i\}_{i=1}^{\infty} \) is a decreasing sequence of open sets in \( X \times Y \) such that \( \bigcap_{i=1}^{\infty} W_i = \emptyset \).

Since \( C \subset (X \times Y) - (P \times Q) = \{(X - P) \times Y\} \cup \{X \times (Y - Q)\} \), \( \phi_i(C) \subset \{h_i(X - P) \times Y\} \cup \{X \times k_i(Y - Q)\} \subset (U_i \times Y) \cup (X \times V_i) = W_i \) for each \( i = 1, 2, \ldots. \)

Hence \( X \times Y \) is \( DCS/(x, y) \) space.

**Theorem 2.10.** If \( P \) is a separable subspace of \( X \) then \( X \) is separable.

**Proof.** Let \( A \) be a countable dense subset of \( P \), and let \( \{h_i\}_{i=1}^{\infty} \) be a sequence of homeomorphisms in \( X \) such that \( \bigcup_{i=1}^{\infty} h_j(P) = X \). Then \( D = \bigcup_{i=1}^{\infty} h_i(A) \) is a countable dense subset of \( X \) since \( h_i(A) \) is a countable dense subset of \( h_i(P) \) for each \( i \).

**References**


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