Let $R$ be a commutative ring with 1. We shall denote the category consisting of all $R$-modules and $R$-homomorphisms by $M(R)$. Then $M(R)$ is a complete and cocomplete category, and also it is a $C_3$-category with a projective generator $R$ ([3], p. 73). Therefore $M(R)$ has enough injectives ([2], p. 262). Let $\text{End}(R)$ be the category of all $R$-endomorphisms, and let $\text{Idm}(R)$ be the full subcategory of $\text{End}(R)$ whose objects are idempotents. In [1], Hou proved that $\text{End}(R)$ and $\text{Idm}(R)$ have enough projectives. Let $\text{Idm-Iso}(R)$ be the full subcategory of $\text{Idm}(R)$ whose objects are isomorphisms. The purpose of this paper is to prove that

(i) $\text{End}(R)$ is complete and cocomplete, and so is $\text{Idm}(R)$ (Theorem 7 and Corollary 8),

(ii) $\text{Idm-Iso}(R)$ has enough injectives (Theorem 11), and to prove a property about $\text{Idm}(R)$ (Proposition 9).

1. Definitions in $\text{End}(R)$

It is well known that for each object $\alpha$ of $\text{End}(R)$ there exists a unique object $A$ of $M(R)$ such that $\alpha$ belongs to $\text{hom}(A, A)$, where $\text{hom}(A, A)$ is an abelian group consisting of all $R$-module homomorphisms from $A$ to itself. A morphism $f: \alpha \to \beta$ in $\text{End}(R)$ is an $R$-module homomorphism such that $f\alpha = \beta f$, where $\alpha \in \text{hom}(A, A)$ and $\beta \in \text{hom}(B, B)$.

**Definition 1.** A morphism $f: \alpha \to \beta$ ($\alpha \in \text{hom}(A, A)$ and $\beta \in \text{hom}(B, B)$) in $\text{End}(R)$ is said to be a monomorphism if $f: A \to B$ is a monomorphism in $M(R)$. Dually, a morphism $f: \alpha \to \beta$ in $\text{End}(R)$ is an epimorphism if $f: A \to B$ is an epimorphism in $M(R)$.

For a morphism $f: \alpha \to \beta (f: A \to B \text{ in } M(R))$ in $\text{End}(R)$ the kernel of $f$ is defined by $\alpha \mid \text{Ker}(f) : \text{Ker}(f) \to \text{Ker}(f)$, where $\text{Ker}(f)$ is the kernel of $f: A \to B$ in $M(R)$. Similarly, the image of $f$ in $\text{End}(R)$ is defined by $\beta \mid \text{Im}(f)$, where $\text{Im}(f)$ is the image of $f: A \to B$ in $M(R)$.

The cokernel $\bar{\beta}: B/\text{Im}(f) \longrightarrow B/\text{Im}(f)$ of $f: \alpha \to \beta$ in $\text{End}(R)$ is defined by
\[ \beta(b + \text{Im}(f)) = \beta(b) + \text{Im}(f) \]

for \( b + \text{Im}(f) \in B/\text{Im}(f) \).

With the above notions we can easily prove that the categories \( \text{End}(R) \), \( \text{Idm}(R) \) and \( \text{Idm-Iso}(R) \) are abelian (\([1]\)).

**Definition 2.** A subobject \( \beta : B \rightarrow B \) in \( M(R) \) of \( \alpha \in \text{End}(R) \) (\( \alpha : A \rightarrow A \) in \( M(R) \)) is an object of \( \text{End}(R) \) satisfying (i) \( B \) is a submodule of \( A \) in \( M(R) \). (ii) \( \beta = \alpha | B \). The intersection of subobjects \( \beta \) and \( \gamma : C \rightarrow C \) in \( M(R) \) is \( \alpha | B \cap C \). It is easy to prove that

**Proposition 3.** For a morphism \( f : \alpha \rightarrow \beta \) the kernel of \( f \) is a subobject of \( \alpha \), and the image of \( f \) is a subobject of \( \beta \).

**Definition 4.** In \( \text{End}(R) \), consider a diagram

\[
\begin{array}{ccc}
\alpha' & \rightarrow & \beta' \\
\downarrow & & \downarrow \\
\alpha & \rightarrow & \beta \\
\end{array}
\]

where \( f \) is any morphism and the vertical morphisms are monomorphisms. In this case, the subobject \( \alpha' \) is said to be carried into the subobject \( \beta' \) by \( f \) if there is a morphism \( \alpha' \rightarrow \beta' \) making the above diagram is commutative.

The union of a family \( \{\alpha_i\}_{i=1} \) of subobjects of an object \( \alpha \) is defined as a subobject \( \alpha' \) of \( \alpha \), denoted by \( \alpha' = \cup \alpha_i \), which is preceded by each of the \( \alpha_i \), and which has the following property: If \( f : \alpha \rightarrow \beta \) and each \( \alpha_i \) is carried into some subobject \( \beta' \) by \( f \), then \( \alpha' \) is also carried into \( \beta' \) by \( f \). In this case, if \( \alpha_i : A_i \rightarrow A_i \) in \( M(R) \), then

\[ \cup \alpha_i : \cup A_i \rightarrow \cup A_i \]

where for \( a_i \in A_i \subseteq \cup A_i \), \( \cup \alpha_i(a_i) = \alpha_i(a_i) \).

**Proposition 5.** In \( \text{End}(R) \), for any direct family \( \{\alpha_i\} \) of subobjects of \( \alpha \) and any subobject \( \beta \) of \( \alpha \)

\[ (\cup \alpha_i) \cap \beta = \cup (\alpha_i \cap \beta). \]

**Proof.** For each \( i \) we put \( \alpha_i : A_i \rightarrow A_i \), where \( A_i \) is a submodule of \( A \) in \( M(R) \). By a direct family \( \{\alpha_i\} \) we mean that \( A_i \cap A_j = \{0\} \), if \( i \neq j \). Then

\[ \bigoplus \alpha_i = \alpha | \bigoplus A_i = \alpha | \cup A_i = \cup \alpha_i. \]

Since \( (\cup A_i) \cap B = \cup (A_i \cap B) \) we have

\[ \alpha | (\cup A_i) \cap B = \alpha | (\cup A_i \cap B) \]
PROPOSITION 6. The category $\text{End}(R)$ has equalizers, and so has the category $\text{Idm}(R)$.

Proof. If $f, g : \alpha \rightarrow \beta$ are morphisms in $\text{End}(R)$, then in $M(R)$ we have the commutative diagrams:

$\begin{array}{c}
A \xrightarrow{f} B \\
\downarrow \alpha \quad \downarrow \beta \\
A \xrightarrow{g} B
\end{array}$

Put $K = \{a \in A | f(a) = g(a)\}$, then $K$ is a submodule of $A$. If we shall define $\gamma : K \rightarrow K$ by $\gamma = \alpha | K$ then the inclusion $K \rightarrow A$ is a morphism $\gamma \rightarrow \alpha$ in $\text{End}(R)$. Then, it is easily proved that $\gamma \rightarrow \alpha$ is an equalizer for $f$ and $g$.

2. Main results

THEOREM 7. $\text{End}(R)$ is complete and cocomplete.

Proof. Let $\{ \beta \rightarrow \alpha_i \}_{i \in I}$ be a compatible family for $D$. Then, for all $m \in M$ and $d(m) = (i, j)$ there is a commutative diagram:

$\begin{array}{c}
\alpha_i \\
\downarrow g_i \\
B \\
\downarrow D(m) = f_{ij} g_i = g_j \\
\downarrow A_j
\end{array}$

such that $\alpha_j f_{ij} = f_{ij} \alpha_i$, $\alpha_i g_i = g_i \beta$ and $\alpha_j g_j = g_j \beta$ where $\beta : B \rightarrow B$ and $\alpha_i : A_i \rightarrow A_i$ in $M(R)$. Therefore we have a compatible family $\{B \rightarrow A_i\}_{i \in I}$ for $D$ in $M(R)$. In this case

$\bigcap_{m \in M} \text{Equ}(P_h, D(m) P_j) \subset \prod_{k \in I} A_h P_k$ $A_i$

is a limit for $D$ in $M(R)$. Let us define $\times A_h \rightarrow \times A_h$ such that $(\times A_h) | A_h = \alpha_h$ where $\alpha_h \in \text{hom}(A_h, A_h)$. Then $\times A_h$ is a product of the family $\{\alpha_i\}_{i \in I}$ in $\text{End}(R)$. Since

$\bigcap_{m \in M} \text{Equ}(P_h, D(m) P_j) \subset \times A_h$ (§3 p. 47)

we can put $\times A_h = \bigcap_{m \in M} \text{Equ}(P_h, D(m) P_j)$

where for all $h \in I$. 

Define

\[ \times (\alpha_k | A_k') : \times A_k' \longrightarrow \times A_k \]

then \( \times (\alpha_k | A_k') \) is a limit for \( D \) in \( \text{End}(R) \) by the following reasons.

At first, we have the commutative diagram for all \( i \in I \):

\[
\begin{array}{ccc}
\times A_k' & \overset{p_i|A_i'}{\longrightarrow} & A_i \\
\downarrow \times (\alpha_k | A_k') & & \downarrow \alpha_i \\
\times A_k & \overset{p_i|A_i}{\longrightarrow} & A_i
\end{array}
\]

Since it is easy to see that \( \alpha_i|\times a_k' = (\times a_k') = (p_i|A_i') (\times (\alpha_k | A_k')) \)
\( (\times a_k') \) for \( \times a_k' \in \times A_k' \), and that \( \{p_i|A_i' : \times (\alpha_k | A_k') \longrightarrow \alpha_i \}_{i \in I} \)

is a compatible family for \( D \) in \( \text{End}(R) \).

Next, for a compatible family \( \{g_i : \beta \longrightarrow \alpha_i\} \) we have a unique \( R \)-module homomorphism \( h : B \longrightarrow \times A_k' \) satisfying the commutative diagram (for all \( i \in I \)):

\[
\begin{array}{ccc}
B & \overset{h}{\longrightarrow} & \times A_k' \\
\downarrow g_i & & \downarrow \beta \\
\times A_k & \overset{h}{\longrightarrow} & \times (\alpha_k | A_k')
\end{array}
\]

in which \( h = \times g_h \), i.e. for all \( b \in B \) \( h(b) = \times g_h(b) \in \times A_k' \). Moreover the diagram

\[
\begin{array}{ccc}
B & \overset{h}{\longrightarrow} & \times A_k' \\
\downarrow \beta & & \downarrow (\times \alpha_k | A_k') \\
B & \overset{h}{\longrightarrow} & \times A_k'
\end{array}
\]

is commutative since \( \alpha_i g_i = g_i \beta \). It means that there exists a unique morp-

hism \( h : \beta \longrightarrow \times (\alpha_k | A_k') \) such that for all \( i \in I \) \( g_i = (p_i|A_i')h \). For a cocomp-
A note on the category $\text{End}(R)$

Atible family $\{\alpha_i \to \gamma\}_{i \in I}$ ($\alpha_i : A_i \to A_i$ and $\gamma : C \to C$ in $M(R)$) we see that $A_i \mathop{\xrightarrow{u_i}} \bigoplus_{h \in I} A_h \mathop{\xrightarrow{\bigcup_{m \in M} (u_k - u_j D(m))}} \bigoplus_{h \in I} \text{Im}(u_k - u_j D(m))$ ([3], p. 47) is a colimit for $D$ in $M(R)$. Put

$$\overline{A}_i = A_i / \bigcup_{m \in M} (u_k - u_j D(m))$$

for all $i \in I(A_i \subset \bigoplus A_i)$. Then

$$\bigoplus_{h \in I} \text{Im}(u_k - u_j D(m)) = \bigoplus_{h \in I} \overline{A}_h$$

Define

$$\bigoplus \bar{a}_h : \bigoplus_{h \in I} \overline{A}_h \to \bigoplus_{h \in I} \overline{A}_h$$

such that for $a_i + \bigcup_{m \in M} (u_k - u_j D(m)) \in \overline{A}_i$

$$(\bigoplus \bar{a}_h) (a_i + \bigcup_{m \in M} (u_k - u_j D(m))) = a_i (a_i) + \bigcup_{m \in M} (u_k - u_j D(m)).$$

Let $\bar{u}_i : A_i \to \bigoplus_{h \in I} \overline{A}_h$ be induced from the $i$th injection $u_i : A_i \to \bigoplus_{h \in I} A_h$ for all $i \in I$. Then we have the commutative diagram

$$\begin{array}{ccc}
A_i & \longrightarrow & \bigoplus_{h \in I} \overline{A}_h \\
\alpha_i \downarrow & & \downarrow \bigoplus \bar{a}_h \\
A_i & \longrightarrow & \bigoplus_{h \in I} \overline{A}_h
\end{array}$$

which means that $\bar{u}_i : \alpha_i \to \bigoplus \bar{a}_h$ is a morphism of $\text{End}(R)$.

We can prove that $\{\alpha_i \to \bigoplus \bar{a}_h\}$ is a colimit for $D$ by the same way as in the above proof with respect to limit. Hence $\text{End}(R)$ is complete and cocomplete.

**COROLLARY 8.** The category $\text{Idm}(R)$ is also complete and cocomplete.

**Proof.** In the category $\text{Idm}(R)$, let $\{\beta \to \alpha_i\}_{i \in I}$ be a compatible family for a diagram over a scheme $\Sigma = (I, M, d)$ where $\alpha_i : A_i \to A_i$ and $\alpha_i^2 = \alpha_i$ in $M(R)$ for all $i \in I$. Then its limit and colimit for $D$ are $\bigotimes_{h \in I} (\alpha_i | A_i')$ and $\bigotimes_{h \in I} \bar{a}_h$, respectively (see proof of Theorem 7), because $$(\bigotimes_{h \in I} (\alpha_i | A_i'))^2 = \bigotimes_{h \in I} (\alpha_i^2 | A_i')$$

$= \bigotimes_{h \in I} \bar{a}_h$ and $$(\bigotimes_{h \in I} \bar{a}_h)^2 = \bigotimes_{h \in I} \bar{a}_h.$$
PROPOSITION 9. In the category $\text{Idm}(R)$, if $\{f_i: \beta \to \alpha_i\}_{i \in I}$ is a compatible family, then so are $\{\alpha_i f_i: \beta \to \alpha_i\}_{i \in I}$ and $\{f_i \beta: \beta \to \alpha_i\}_{i \in I}$.

Proof. For $i \in I$ we assume that $\alpha_i: A_i \to A_i$ and $\beta: B \to B$ in $\text{M}(R)$. For each $(i, j) \in I \times I$ we have to verify that the diagrams

\[
\begin{array}{ccc}
B & \xrightarrow{\alpha_i f_i} & A_i \\
\downarrow{\beta} & & \downarrow{\alpha_i} \\
B & \xrightarrow{\alpha_i f_i} & A_i \\
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
B & \xrightarrow{\alpha_i f_i} & A_i \\
\downarrow{f_{ij}} & & \downarrow{\alpha_i f_i} \\
A_j & \xrightarrow{\alpha_i f_i} & A_i \\
\end{array}
\]

are commutative. Since $\alpha_i^2 = \alpha_i$, $\beta = \beta$, $f_{ij} f_i = f_{ij}$, $f_i \beta = \alpha_i f_i$ and $\alpha_i f_{ij} = f_{ij} \alpha_i$ we have

$\alpha_i \alpha_i f_i = \alpha_i f_i \beta = \alpha_i f_i$ and $f_{ij} \alpha_i f_i = \alpha_i f_{ij}$. 

For $\{f_i \beta: \beta \to \alpha_i\}$ we can prove it by the same way as above.

PROPOSITION 10. If $\alpha \in \text{Idm-Iso}(R)$ then $\alpha$ is an identity map.

Proof. By the definition of $\alpha$ we have $\alpha^2 = \alpha$. Since $\alpha$ is an isomorphism there exists the inverse $\alpha^{-1}$ of $\alpha$. Thus $\alpha^{-1} \alpha^2 = \alpha^{-1} \alpha = 1$.

THEOREM 11. The category $\text{Idm-Iso}(R)$ has enough injectives.

Proof. Noting that the category $\text{M}(R)$ has enough injectives, it suffices to prove that $\text{M}(R)$ and $\text{Idm-Iso}(R)$ are isomorphic.

By Proposition 9 the object class of $\text{Idm-Iso}(R)$ is the class $\{1_A | A \in \text{M}(R)\}$.

For $1_A$ and $1_B$ in $\text{Idm-Iso}(R)$ it is easily see that $\text{hom}(1_A, 1_B) = \text{hom}(A, B)$, where $\text{hom}(1_A, 1_B)$ is the $R$-module of all morphisms from $1_A$ to $1_B$ in $\text{Idm-Iso}(R)$. Therefore the functor $F: \text{M}(R) \to \text{Idm-Iso}(R)$ defined by $F(A) = 1_A$ for $A \in \text{M}(R)$ is an isomorphism. Thus $\text{Idm-Iso}(R)$ has enough injectives.

NOTE: It is well known that $\text{M}(R)$ has a generator $R$. But $1_R$ is not a generator of $\text{End}(R)$ and $\text{Idm}(R)$. If $f: \alpha \to \beta$ ($\neq 0$) is a morphism in $\text{End}(R)$ or $\text{Idm}(R)$ we can not insure existence of a morphism $g: 1_R \to \alpha$ such that $fg \neq 0$ and $\alpha g = g$. Even if $\text{End}(R)$ and $\text{Idm}(R)$ are $C_3$-categories (by Proposition 5, Theorem 7 and corollary 8), maybe they do not have enough injectives.

References


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