LOCALLY ORDER-CONVEX SPACES.

By V. Murali

0. Abstract

The first part of this note is concerned with a neighbourhood base characterisation of locally order-convex spaces. The notions of order-$*$-inductive limits and order ultrabornologicity in the class of locally order-convex spaces are introduced and studied in the latter part. These are the non-convex generalisation of $o$-inductive limits and $o$-bornological spaces.

1. Introduction

A locally order-convex space is a partially ordered vector space together with a linear topology for which there exists a base of neighbourhoods of the origin consisting of order-convex and balanced subsets. Kist [3] studied locally $o$-convex (order-convex and convex) spaces. Iyahen [2] developed concepts of $*$-inductive limits and ultrabornological spaces in the general topological vector spaces setting. The objects of this note are to define and give some results on order $*$-inductive limits and on order ultrabornological properties of locally order-convex spaces, using analogous techniques of Iyahen [2] (thus generalising the results of Kist [3]).

In section 2, we define an analogue of a suprabarrel in a topological vector space and use it to prove some basic results on locally order-convex spaces. Section 3 is devoted to the results on the finest locally order-convex topologies making certain positive linear mappings continuous (that is, order $*$-inductive limits) and on the finest locally order-convex topologies on a partially ordered vector spaces. In the last section, we study those locally order-convex spaces $E$ which have the property that every positive linear mapping of $E$ with range in any locally order-convex space is continuous.

Regarding the theory of topological vector spaces we refer to Horvath [1] and for those undefined order-theoretic terms we refer to Schaefer [5].

2. Locally order-convex spaces

DEFINITION 2.1. A subset $U$ of a partially ordered vector space $(E, C)$ is called
an order-convex suprabarrel (hereafter abbreviated to o-suprabarrel) if \( U \) is balanced, absorbent and order-convex and, if there exists a sequence \( (U_n) \) of balanced, absorbent and order-convex subsets of \( E \) such that \( U_1 + U_1 \subseteq U \) and \( U_{n+1} + U_{n+1} \subseteq U_n \) for all \( n \). If, in addition, \( U \) is closed we call it an order-convex ultrabarrel (o-ulrabarrel). We call \( (U_n)(n=1,2,\ldots) \) a defining sequence for \( U \).

It is immediate that balanced, absorbent, o-convex subsets of a partially ordered vector space \( (E, C) \) are o-suprabarrels. However, an o-suprabarrel need not be o-convex and need not have a defining sequence of o-convex sets. Intersection of a finite number of o-suprabarrels is an o-suprabarrel. If \( U \) is a suprabarrel in \( (E, C) \), then \([U]\), the order-convex hull of \( U \) is an o-suprabarrel. The inverse image of an o-suprabarrel by a positive linear mapping is an o-suprabarrel. The image of an o-suprabarrel by a positive linear mapping is an o-suprabarrel provided the mapping is onto.

**DEFINITION 2.2.** An \( F \)-semi-norm \( \nu \) on a partially ordered vector space \( (E, C) \) is called monotone if \( \nu(x) \leq \nu(y) \) whenever \( 0 \leq x \leq y \) in \( E \).

Let \( U \) be an o-suprabarrel with a defining sequence \( (U_n) \) \( (n=1,2,\ldots) \) in a partially ordered vector space \( (E, C) \). By the method of construction on page 3 of Wealbroeck [6], we can associate an \( F \)-semi-norm \( \nu \) with \( U \), as follows:

\[
\nu(y) = \inf\{\beta : y \in W_\beta, \ (y \in E) \}
\]

where \( W_\beta = E \) for \( \beta \geq 1 \) and \( W_\beta = \sum_{t=1}^{n} U_k \) for every dyadic rational \( \beta = \sum_{k=1}^{n} t_k 2^{-k} \). Suppose \( E \) has the decomposition property, then \( W_\beta \) is order-convex, as it is the sum of order-convex sets \( U_k \). Hence \( y \in W_\beta \) and \( 0 \leq x \leq y \) imply \( x \in W_\beta \). That is, \( \nu(x) \leq \nu(y) \) whenever \( 0 \leq x \leq y \).

Thus an \( F \)-semi-norm associated with an o-suprabarrel is monotone provided \( E \) has the decomposition property.

**DEFINITION 2.3.** (Wong and Ng [7]) A linear topology \( \tau \) on a partially ordered vector space \( (E, C) \) is said to be locally order-convex if it admits a neighbourhood base at 0 consisting of order-convex sets; in this case, we shall say \( (E, C, \tau) \) is a locally order-convex space.

**REMARK.** Locally order-convex topologies were first considered by Namioka [4]. He called these topologies locally full.

The following proposition gives an useful characterisation of a neighbourhood base at the origin in a locally order-convex space.

**PROPOSITION 2.4.** In a locally order-convex space, there exists a base of
neighbourhoods at origin consisting of $o$-suprabarrels; Conversely, let $(E,C)$ be a partially ordered vector space and let $\mathcal{W}$ be a filter base at the origin consisting of $o$-suprabarrels with their defining sequences. Then there exists a unique vector topology $\tau$ on $E$ for which $E$ is a locally order-convex space and for which $\mathcal{W}$ is a base of $\tau$-neighbourhoods at the origin.

Proof is straightforward.

**COROLLARY 2.5.** The collection of all $o$-suprabarrels in a partially ordered vector space $(E,C)$ is a neighbourhood base for the finest locally order-convex topology on $E$.

For certain class of partially ordered topological vector spaces, the notion of locally order-convexity is equivalent to a condition in terms of continuous $F$-semi-norms. We shall make this precise in the next theorem, but first we require a lemma due to Namioka [4, p.19].

**LEMMA 2.6.** Let $(E,C,\tau)$ be a partially ordered topological vector space. Then the following are equivalent.

1. The space $(E,C,\tau)$ is locally order-convex;
2. Given a $\tau$-neighbourhood $U$ of zero, there exists a $\tau$-neighbourhood $V$ of zero such that $0 \leq x \leq y$ for some $y$ in $V$ implies $x \in U$.

**THEOREM 2.7.** Let $(E,C,\tau)$ be a partially ordered topological vector space with decomposition property. Then the following statements are equivalent.

1. $\tau$ is a locally order-convex topology.
2. The family of all $\tau$-continuous monotone $F$-semi-norms determines the topology $\tau$.

**PROOF.** (1)$\Rightarrow$(2). Let $\{\nu_i\}$ $(i \in I)$ be the family of all $\tau$-continuous monotone $F$-semi-norms, and $\tau'$ be the topology generated by $\{\nu_i\}$ $(i \in I)$. It is easy to see that $\tau'$ is coarser than $\tau$. We now show that $\tau$ is coarser than $\tau'$. Let $U$ be a balanced $\tau$-neighbourhood of the origin. Since $\tau$ is locally order-convex, there exists an $o$-suprabarrel $V$ contained in $U$. By the remark preceding definition 2.3, the $F$-semi-norm $\nu_V$ of $V$, is $\tau$-continuous and monotone. Also, $\{x \in E : \nu_V(x) < 1\} \subset V \subset U$; so $U$ is a $\tau'$-neighbourhood, as required.

(2)$\Rightarrow$(1). Let $U$ be a $\tau$-neighbourhood of 0. Then there exists a finite number $\{\nu_i\}$ $(i=1, 2, \ldots, n)$ of monotone $F$-semi-norms such that $V = \{x \in E : \max_{i=1,2,\ldots,n} \nu_i(x) < \varepsilon : 0 < \varepsilon < 1\} \subset U$. $V$ satisfies the property (2) of Lemma 2.6.
and so \((E,C,\tau)\) is locally order-convex.

It is useful to note that there is a method available, for constructing locally order-convex topologies from vector topologies on partially ordered vector spaces. We shall not describe it here but refer to [7, p.56].

We conclude this section with a remark on the finest locally order-convex topology. Let \(e\) be an order-unit in a partially ordered vector space \((E,C)\). Then the set \([-e,e]\) is balanced, convex, and absorbing. Hence the Minkowski functional \(\nu_e\) of \([-e,e]\) is a semi-norm on \(E\). Kist in [3] observed that the topology \(\tau_e\) induced by \(\nu_e\) on \(E\) is the finest locally \(\sigma\)-convex topology. We claim that \(\tau_e\) coincides with the finest locally order-convex topology \(\tau\) on \(E\); in fact, if \(V\) is any balanced order-convex \(\tau\)-neighbourhood of the origin in \(E\), then \(V\) is absorbing. So there exists a \(\lambda>0\) such that \(\lambda e \in V\), implying \(\lambda [-e,e] \supset V\) Hence \(V\) is a \(\tau_e\)-neighbourhood.

3. Order \(\ast\)-inductive limits

Let \((E,C)\) be a partially ordered vector space, and \((E_i,C_i,\tau_i)\) a family of locally order-convex spaces, \((i \in I)\). Let \(f_i\) be a positive linear mapping from \(E_i\) into \(E\) for each \(i \in I\). Then the order \(\ast\)-inductive limit (hereafter abbreviated to \(\sigma\)-\(\ast\)-inductive limit) topology on \(E\) with respect to the family \((E_i,C_i,\tau_i):f_i\) is defined to be the finest locally order-convex topology on \(E\) for which all the positive linear mapping \(f_i\)'s are continuous.

PROPOSITION 3.1. \(\tau\) always exists on \(E\).

PROOF. Let \(\mathcal{L}\) be the set of all locally order-convex topologies on \(E\). The topology \(\eta = \{\phi, E\}\) is locally order-convex and is the least element of \(\mathcal{L}\). Since finite intersections of \(\sigma\)-suprabarrels is an \(\sigma\)-suprabarrel, the supremum of an arbitrary non-empty family of locally order-convex topologies is again locally order-convex. Let \(\mathcal{L}_0\) be the subset of \(\mathcal{L}\) consisting of those topologies for which each positive linear mapping \(f_i\) is continuous. \(\mathcal{L}_0\) is non-empty since \(\eta \in \mathcal{L}_0\) and the supremum of \(\mathcal{L}_0\) which also belongs to \(\mathcal{L}_0\) is obviously the required topology.

The space \(E\) equipped with the \(\sigma\)-\(\ast\)-inductive limit topology is called the \(\sigma\)-\(\ast\)-inductive limit.

We observe that the \(\sigma\)-\(\ast\)-inductive limit topology on \(E\) is a linear topology, and so weaker than the strongest linear topology on \(E\) relative to which all
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the $f_i$'s are continuous, that is, the linear *-inductive limit topology as defined in [2, p. 286]. Thus the two topologies coincide if and only if, the linear *-inductive limit topology on $E$ is locally order-convex. At present, we do not have an example to show that the two topologies are distinct.

**Proposition 3.2.** Let $(E_i, C_i, \tau_i)$ $(i \in I)$ be a family of locally order-convex spaces; For each $i \in I$, let $f_i$ be a positive linear mapping of $E_i$ into a partially ordered vector space $(E, C)$. Let $Z = \{U\}$ be the collection of all o-suprabarrels of $E$ with the property that, for each $i \in I$, $f_i^{-1}(U)$, $f_i^{-1}(U_n)$ $(n=1, 2, \ldots)$ are $\tau_i$-neighbourhoods of 0 in $E_i$, where $U_n$ is a defining sequence of $U$. Then $Z$ is a base of neighbourhoods of 0 in $E$ for the o-*$-inductive limit topology with respect to the locally order-convex spaces $(E_i)$ and the positive linear mappings $(f_i)$.

**Proof.** Clearly, $Z$ forms a base of neighbourhoods of 0 in $E$ for a locally order-convex topology $\tau'$ on $E$, by proposition 2.4. If $W$ is a base of balanced, order-convex neighbourhoods of 0 for any other locally order-convex topology $\tau''$ on $E$ for which all the $f_i$'s are continuous, then each $W \in W'$ is absorbing, o-suprabarrel in $E$. It is straightforward to check that $W \in Z$, and so $W \subseteq Z$ from which it follows that $\tau'' \subseteq \tau'$. Thus $\tau'$ is the strongest such topology and therefore $\tau'$ is the o-*$-inductive limit topology on $E$.

**Corollary 3.3.** If $(F, \gamma)$ is a locally order-convex space and if $g$ is a positive linear mapping of $E$ into $F$, then $g$ is continuous with respect to the o-*$-inductive limit topology $\tau$ on $E$ if and only if, $g \circ f_i$ is continuous for each $i \in I$.

**Proof.** If $g$ is continuous, then clearly the mappings $g \circ f_i$ are all continuous. Conversely, suppose $g$ is a positive linear mapping such that $g \circ f_i$ is continuous for each $i \in I$. Let $U_0$ be any balanced order-convex $\eta$-neighbourhood of the origin in $F$. Choose a sequence of balanced, order-convex $\gamma$-neighbourhoods $\{U_n\}$ $(n=1, 2, \ldots)$ such that $U_n + U_n \subseteq U_{n-1}$, $(n=1, 2, \ldots)$. Then $g^{-1}(U_0)$ is a balanced, absorbing o-suprabarrel in $E$, with a defining sequence $\{g^{-1}(U_n)\}$ $(n=1, 2, \ldots)$. Also $f_i^{-1}(g^{-1}(U_n)) = (g \circ f_i)^{-1}(U_n)$ is a $\tau_i$-neighbourhood of 0 in $E_i$ for each $i \in I$ and $n=0, 1, 2, \ldots$ Thus, by proposition 2.3, $g^{-1}(U_0)$ is a $\tau$-neighbourhood of 0 in $E$ and so $g$ is continuous.

Let $(E, C)$ be a partially ordered vector space. For each $a \in E$ with $a \geq 0$, let 

$$E_a = \bigcup_{i \in I} \{x : x \in E, -la \leq x \leq la\}$$


\[ C_a = E_a \cap C. \]

Then \( E_a \) is a subspace and \( a \) is an order-unit for \( (E_a, C_a) \). Let \( \tau_a \) be the locally order-convex topology on \( E_a \) induced by the semi-norm \( \nu_a \) of \([-a,a] \) in \( E_a \).

Then we have the following analogue of proposition 5.2 [3].

**Theorem 3.4.** Let \( (E, C, \tau) \) be a locally order-convex space. Then \( \tau \) is the finest order-convex topology on \( E \) if and only if \( (E, C, \tau) \) is the \( o \)-*-inductive limit of \( (E_i, C_i, \tau_i) \) for \( a \in E \), with respect to inclusion mappings \( i_a \).

**Proof.** Necessity. Let \( V \) be any balanced, order-convex \( \tau \)-neighbourhood of \( 0 \) in \( (E, C) \). Then \( V \) is an \( o \)-superbarrel with a defining sequence \( (V_n) \) (\( n=1, 2\ldots \)), say. The sets \( V \cap E_a \), \( V_n \cap E_a \) for each \( a \in E \), \( n=1, 2\ldots \), are balanced, order-convex and absorbing. Moreover \( V_n \cap E_a + V_n \cap E_a \subseteq V_{n-1} \cap E_a \) (\( n=2, 3\ldots \)), and \( V_1 \cap E_a + V_1 \cap E_a \subseteq V \cap E_a \) for each \( a \in E \). That is \( V \cap E_a \) is an \( o \)-superbarrel in \( E_a \) for each \( a \in E \). Since \( \tau_a \) is the finest locally order-convex topology in \( E_a \), \( V \cap E_a \) are \( \tau_a \)-neighbourhoods of origin in \( E_a \). Hence \( V \) is an \( o \)-*-inductive limit neighbourhood, by proposition 3.2.

Sufficiency. Let \( V \) be any \( o \)-superbarrel in \( (E, C) \), with a defining sequence \( (V_n) \). Then it is obvious that \( i_a^{-1}(V) = V \cap E_a \) (\( i_a^{-1}(V_n) = V_n \cap E_a \) (\( n=1, 2\ldots \))) is an \( o \)-superbarrel in \( E_a \) for \( a \in E \), and so a \( \tau_a \)-neighbourhood in \( E_a \). Also for each \( n=1, 2\ldots \), \( V_n \cap E_a \) is an \( o \)-superbarrel in \( E_a \) and hence a \( \tau_a \)-neighbourhood.

The proposition 3.2, now implies \( V \) is a neighbourhood in the order \( \ast \)-inductive limit topology. Therefore, the order \( \ast \)-inductive limit topology coincides with the finest order-convex topology by Corollary 2.5.

We conclude this section with an useful analogue of proposition 2.2 of [2].

**Theorem 3.5.** Let \( (E, C, \tau) \) be the order \( \ast \)-inductive limit of a family of locally order-convex spaces \( (E_i, C_i, \tau_i) \) (\( i \in I \)) with respect to positive linear mappings \( (f_i) \). For each \( i \in I \), let \( V_i \) be a balanced, order-convex \( \tau_i \)-neighbourhood of \( 0 \) in \( E_i \), and let \( U \) be the order-convex hull of \( \bigcup_{\emptyset \neq i \in \Phi} f_i(V_i) \) the union being taken over all finite subsets \( \Phi \) of \( I \). Then \( U \) is a \( \tau \)-neighbourhood of \( 0 \) in \( E \).

If \( I \) is countable, then as \( V_i \) runs through a base of balanced, order-convex \( \tau_i \)-neighbourhoods of \( 0 \) in \( E_i \), the order-convex hull of the above sets form a base of \( \tau \)-neighbourhoods of \( 0 \) in \( E \).

**Proof.** Let \( U = \bigcup_{\emptyset \neq i \in \Phi} f_i(V_i) \) as given in From Iyahan [2], we know that
$U$ is a neighbourhood in the $*$-inductive limit topology $\eta$. Since the order-convex hull $[U]$ of $U$ is a neighbourhood in the finest order-convex topology coarser than $\eta$, it follows $[U]$ is a $\tau$-neighbourhood. Similarly, the second part of the theorem follows from the corresponding part of proposition 2.2 of [2].

Since the $*$-inductive limit of a sequence of locally convex spaces is locally convex, and the order-convex hull of a convex set is $\sigma$-convex (see Kist [3]), we have the following:

**Corollary 3.6.** The $o$-$\alpha$-inductive limit of a sequence of locally $\sigma$-convex spaces, is locally $\sigma$-convex, and thus coincides with the $o$-inductive limit. (See Kist [3]).

### 4. $o$-ultrabornological spaces

**Definition 4.1.** A locally order-convex space $E$ is called $o$-ultrabornological ($o$-ultrabornological) if every bounded positive linear mapping from $E$ into any locally order-convex space is continuous.

We conjecture that the attributes of ultrabornological and $o$-ultrabornological are distinct when applied to the class of locally order-convex space. But we are unable to substantiate this. Also, at present, we do not know whether an $o$-bornological space as defined by Kist [3], is $o$-ultrabornological or not.

However, the class of $o$-ultrabornological space is non-empty, as it contains metrisable locally order-convex spaces. In particular, if the topology $\tau$ of a partially ordered topological vector space $(E,C)$ is given by a single monotone $F$-semi-norm, then $(E,C,\tau)$ is $o$-ultrabornological.

The following concept is important in the study of $o$-ultrabornological space.

**Definition 4.2.** A subset $B$ of a partially ordered topological vector space $(E,C,\tau)$ is called a bornivorous $o$-suprabarrel if $B$ is a balanced, bornivorous, order-convex subset of $E$ and if there exists a sequence $(B_n)$ of balanced, bornivorous, order-convex subsets of $E$ such that $B_1+B_1 \subseteq B$ and $B_{n+1}+B_{n+1} \subseteq B_n$ for $n=1,2,\ldots$.

The next theorem gives the connection between $o$-ultrabornological spaces and bornivorous $o$-suprabarrel subsets.

**Theorem 4.3.** Let $\tau_1$ be a locally-order-convex topology on a partially ordered vector space $(E,C)$. Then

1. The family of all bornivorous $o$-suprabarrels in $(E,C,\tau_1)$ is a base of neighbourhoods of $0$ for a finer locally order-convex topology $\tau_2$ on $E$. 

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2. The topologies $\tau_1$ and $\tau_2$ have the same bounded subsets.
3. The space $(E, C, \tau_1)$ is o-ultrabornological if and only if $\tau_1 = \tau_2$.
4. and this is so, if and only if every bornivorous o-suprabarrel in $(E, \tau_1)$ is a $\tau_1$-neighbourhood of origin.

The proofs are straightforward.

With some simple modifications of the proofs of proposition 4.1 and Theorem 4.1 of Iyahen [2], we obtain the following analogues.

**PROPOSITION 4.4.** A set of positive linear mappings form an o-ultrabornological space into a locally order-convex space is equicontinuous provided that it is uniformly bounded on bounded sets.

**THEOREM 4.5.** Any o-$\infty$-inductive limit of o-ultrabornological spaces is o-ultrabornological.

By some easy calculations, we can prove the following corollaries of Theorem 4.5.

**COROLLARY 4.6.** If $F$ is a closed subspace of an o-ultrabornological space $E$, then $E/F$ is o-ultrabornological.

**COROLLARY 4.7.** If $f$ is a continuous, open, positive linear mapping of an o-ultrabornological space $E$ onto a locally order-convex space $F$, then $F$ is o-ultrabornological.

**COROLLARY 4.8.** Any countable o-inductive limit of locally o-convex o-ultrabornological spaces is o-ultrabornological.

**DEFINITION 4.9.** A subset $A$ of a linear space is called semi-convex if there is some $\lambda \geq 0$ for which $A + \lambda A$.

**DEFINITION 4.10.** We say that a partially ordered topological vector space is almost order-convex if every bounded subset is contained in some bounded set which is closed, balanced, semi-convex and order-convex.

Clearly every locally o-convex space is almost order-convex and so is any partially ordered topological vector space whose topology is given by a bounded, order-convex neighbourhood of the origin.

The next theorem, an analogue of proposition 6.3 (e) of Kist [3], is a partial converse of theorem 4.5.

**THEOREM 4.11.** Let $(E, C, \tau)$ be an almost order-convex o-ultrabornological
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space, and let \( \mathcal{U} \) be the class of all closed, bounded, semi-convex, balanced, and order-convex subsets of \( E \). For each \( B \in \mathcal{U} \), let \( E_B \) be the linear subspace generated by \( B \). Then

1. \( E_B \) is a partially ordered vector space.

2. There exists a \( p \)-normed, locally order-convex topology \( \tau_B \) on \( E_B \), for a suitable \( 0 < p \leq 1 \).

3. \( (E, C, \tau) \) is the \( o \)-inductive limit of \( (E_B, \tau_B) (B \in \mathcal{U}) \) with respect to the inclusion mappings \( i_B \).

PROOF. 1. Take \( C_B = C \cap E_B \) as the positive cone of \( E_B \).

2. Since \( B \) is balanced and semi-convex, there exists a \( \lambda \geq 2 \) such that \( B + B \subseteq \lambda B \). Put \( p = \log 2 / \log \lambda \) and for \( x \in E_B \), define \( \nu_B(x) = \inf(\lambda^p : x \in \lambda B) \). It is easy to check that \( \nu_B \) is a \( p \)-norm on \( E_B \). The topology \( \tau_B \) given by \( \nu_B \) is the required topology.

3. Let \( U \) be a \( \tau \)-neighbourhood of \( 0 \) in \( E \). Then for each \( B \in \mathcal{U} \), \( \lambda B \subseteq U \) for some \( \lambda > 0 \) implying \( \lambda B \subseteq U \cap E_B \). So \( i_B : (E_B, \tau_B) \longrightarrow (E, \tau) \) is continuous for each \( B \in \mathcal{U} \). Moreover, let \( \tau_0 \) be any locally order-convex topology on \( E \) for which each \( i_B \) is continuous. Then it is not difficult to show that \( \tau_0 \) coincides with \( \tau \) as \( (E, \tau) \) is almost order-convex \( o \)-ultrabornological. This completes the proof.

Department of Pure Mathematics,
University College of Wales, Aberystwyth,
Wales, U. K.

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