COVERING CHARACTERIZATION OF LOCALLY UNIFORM SPACES.

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Abstract

In this paper locally uniform spaces have been characterized through covers. It has been shown that both the approaches are equivalent.

1. Introduction

The systematic study of locally uniform spaces has been initiated by Williams [3]. A locally uniform space \((X, \mathcal{D})\) is one which is obtained from a uniform space by localizing the triangle inequality. In this paper we develop the study of these spaces through the covering approach as has been done for uniform spaces by Tukey [1]. Tukey defined a uniform space \((X, \mu)\) as an ordered pair consisting of a non-empty set \(X\) and a collection \(\mu\) of all covers of \(X\) which satisfy the following properties:

(i) \(U_1, U_2 \in \mu\) implies there exists a \(U_3 \in \mu\) such that \(U_3 \prec U_1\) and \(U_3 \prec U_2\).

(ii) if \(U \prec \mathcal{V}\) and \(U \in \mu\) implies \(\mathcal{V} \in \mu\).

When not otherwise specified, the terminology used in this paper is that of Willard [2]. Now we define some notions we have used in the text.

**Definition 1.1.** Let \(X\) be any non-empty set and suppose \(\mathcal{U}\) and \(\mathcal{V}\) are any covers of \(X\). Then \(\mathcal{U}\) is called a refinement of \(\mathcal{V}\), symbolically \(\mathcal{U} \prec \mathcal{V}\), if each member of \(\mathcal{U}\) is contained in some member of \(\mathcal{V}\). Next let \(A \subseteq X\) be any subset of \(X\). The star of \(A\) with respect to \(\mathcal{V}\) is the set \(\text{St}(A, \mathcal{V}) = \bigcup \{U \in \mathcal{V} : U \cap A \neq \emptyset\}\). \(\mathcal{U}\) is said to be a star refinement of \(\mathcal{V}\), written as \(\mathcal{U}^* \prec \mathcal{V}\) if \(\text{St}(U, \mathcal{V}) : U \in \mathcal{U}\) \(\prec \mathcal{V}\). \(\mathcal{U}\) is called a barycentric refinement of \(\mathcal{V}\), written as \(\mathcal{U}^b \prec \mathcal{V}\), if \(\text{St}(x, \mathcal{U}) : x \in X\) \(\prec \mathcal{V}\), where \(\text{St}(x, \mathcal{U}) = \bigcup \{U \in \mathcal{U} : x \in U\}\).

**Definition 1.2.** [3]. Let \(X\) be any non-empty set and let \(\mathcal{D} \subseteq P(X \times X)\) be a subcollection of power set of \(X \times X\). \(\mathcal{D}\) is called a local uniformity on \(X\) if the following axioms are satisfied:

(i) \(\Delta \in \mathcal{D}\) for each \(D \in \mathcal{D}\) where \(\Delta = \{(x, x) : x \in X\}\).

(ii) \(D \in \mathcal{D} \Rightarrow D^{-1} \in \mathcal{D}\) for each \(D \in \mathcal{D}\) where \(D^{-1} = \{(x, y) : (y, x) \in D\}\).
(iii) $D_1, D_2 \in \mathcal{D} \Rightarrow D_1 \cap D_2 \in \mathcal{D}$
(iv) $D \in \mathcal{D}$ and $D \subseteq C \times X \times X \Rightarrow E \in \mathcal{D}$.

(v) for each $x \in X$ and for each $D \in \mathcal{D}$ there exists an $E \in \mathcal{D}$ such that $E \circ E [x] \subseteq D [x]$ where $E \circ E = \{(x, y) :$ for some $z \in X, (x, z) \in E$ and $(z, y) \in E\}$.

The ordered pair $(X, \mathcal{D})$ is called a locally uniform space.

Also a subcollection $\mathcal{A} \subseteq \mathcal{P}(X \times X)$ is called a base for some local uniformity on $X$ if and only if $\mathcal{A}$ satisfies (i), (ii), (iii) and (v) above.

2. Covering locally uniform spaces

**DEFINITION 2.1.** A cover $\mathcal{U}$ of a locally uniform space $(X, \mathcal{D})$ is called a locally uniform cover if and only if it is refined by a cover of the form

$\mathcal{U}_D = \{D[x] : x \in X\}$

for some $D \in \mathcal{D}$.

**THEOREM 2.2.** Let $\mu$ be the collection of all locally uniform covers of a locally uniform space $(X, \mathcal{D})$ then

(i) $\mathcal{U}_1, \mathcal{U}_2 \in \mu$ implies there exists $\mathcal{U}_3 \in \mu$ such that $\mathcal{U}_3 \triangleleft \mathcal{U}_1$ and $\mathcal{U}_3 \triangleleft \mathcal{U}_2$.

(ii) If $\mathcal{U} \triangleleft \mathcal{V}$, $\mathcal{V} \in \mu$, then $\mathcal{V} \in \mu$.

**PROOF.** Let $\mu$ and $(X, \mathcal{D})$ be as above. Let $\mathcal{U}_1, \mathcal{U}_2 \in \mu$. There exist, therefore, $D_1, D_2 \in \mathcal{D}$ such that $\mathcal{U}_1 \triangleleft D_1, \mathcal{U}_2 \triangleleft D_2$. Since $D_1, D_2 \in \mathcal{D}$, we have $D_1 \cap D_2 \in \mathcal{D}$.

Now $(X, \mathcal{D})$ being locally uniform, for each $x \in X$ and $D_1 \cap D_2 \in \mathcal{D}$, there exists a symmetric $D \in \mathcal{D}$ such that $D \circ D [x] \subseteq D_1 \cap D_2 [x]$. We claim $\mathcal{U}_D \triangleleft \mathcal{U}_{D_1}$ and $\mathcal{U}_D \triangleleft \mathcal{U}_{D_2}$. For it, it is sufficient to show that

$St (x, \mathcal{U}_D) \subseteq D_1 [x] \cap D_2 [x]$.

Let,

$y \in St (x, \mathcal{U}_D) = \cup D [z]$

$x \in D [z]$

$z \in X$

$\Rightarrow$ for some $z \in X, y \in D [z]$ and $x \in D [z]$

$\Rightarrow (x, y) \in D, (x, x) \in D$

$\Rightarrow (x, y) \in D \circ D$

$\Rightarrow y \in D \circ D [x] \subseteq D_1 [x] \cap D_2 [x]$

Therefore,

$St (x, \mathcal{U}_D) \subseteq D_1 [x] \cap \mathcal{U}_{D_1} \subseteq \mathcal{U}_1$ implying that $\mathcal{U}_D \triangleleft \mathcal{U}_{D_1}$. 

Similarly we have $\mathcal{U}_D \prec \mathcal{U}_D$. Also $\mathcal{U}_D$ is obviously in $\mu$, let it be denoted by $\mathcal{U}_3$ and thus (i) is proved. (ii) part follows directly from the definition of locally uniform covers.

The converse of the above theorem is true. We state it as follows:

**Theorem 2.3.** Let $\mu$ be a family of covers of a non-empty set $X$ satisfying:

(i) and (ii) of Theorem 2.1. Define for each

$$D^*_\mathcal{Y} = \bigcup \{ u \times u : u \in \mathcal{Y} \}.$$

Then the collection $\mathcal{B} = \{ D^*_\mathcal{Y} : \mathcal{Y} \in \mu \}$ forms a base for a local uniformity on $X$ whose local uniform covers are precisely the members of $\mu$.

**Proof.** Obviously each member of $\mathcal{B}$ contains $A$ and each member is symmetric. Next let $D^*_\mathcal{Y}, D^*_\mathcal{W} \in \mathcal{B}$ then $\mathcal{Y}, \mathcal{W} \in \mu$ and hence there exists $\mathcal{U} \in \mu$ such that $\mathcal{Y} \prec \mathcal{U}, \mathcal{W} \prec \mathcal{U}$. Obviously, then $D^*_\mathcal{Y} \cap D^*_\mathcal{W}$. Finally we prove the local property. Let $D^*_\mathcal{Y} \in \mathcal{B}$ and $x \in X$. $D^*_\mathcal{Y} \in \mathcal{B}$ implies $\mathcal{Y} \in \mathcal{B}$ and hence there exists $\mathcal{W} \in \mu$ such that $\mathcal{Y} \prec \mathcal{W}$. We claim that $D^*_\mathcal{Y} \cap D^*_\mathcal{W} \subseteq D^*_\mathcal{Y} \cap D^*_\mathcal{W}$. Let $y \in D^*_\mathcal{Y} \cap D^*_\mathcal{W}$ then $(x, y) \in D^*_\mathcal{Y} \cap D^*_\mathcal{W}$ and so for some $z \in X$, $(x, z) \in D^*_\mathcal{Y}$ and $(z, y) \in D^*_\mathcal{W}$ which yield $x, z \in V$ for some $V \in \mathcal{Y}$ and $z, y \in W$ for some $W \in \mathcal{W}$. Now we have $\text{St}(z, V) \subseteq \mathcal{Y}$ for some $u \in \mathcal{Y}$. Hence $x, y \in \mathcal{Y}$ implying that $(x, y) \in D^*_\mathcal{Y}$ or $y \in D^*_\mathcal{W}$ whence $D^*_\mathcal{Y} \cap D^*_\mathcal{W} \subseteq D^*_\mathcal{Y} \cap D^*_\mathcal{W}$. Hence $D^*_\mathcal{Y}$ is indeed a base for some local uniformity say $\mathcal{G}$ on $X$. In the final we show that the family of local uniform covers with respect to $\mathcal{G}$ is precisely $\mu$. Let $\eta$ denote the collection of all locally uniform covers with respect to $\mathcal{G}$. We show that $\eta = \mu$.

Observe that $\mu = \eta$ by the assumption of $\mu$, for, if $\mathcal{U} \in \mu$ then by (i) there exists $\mathcal{U}_1 \in \mu$ such that $\mathcal{U}_1 \prec \mathcal{U}$. Then by definition $D^*_\mathcal{U}_1 \subseteq \mathcal{U}$ and $\{ D^*_\mathcal{U}_1 \} \subseteq \mathcal{U}$ whence $D^*_\mathcal{Y} \subseteq \mathcal{Y}$. To show the otherway inclusion, let $\mathcal{G} \in \eta$ then there exists $L \subseteq \mathcal{E}$ such that $L \prec \mathcal{G}$. Now $L \subseteq \mathcal{E}$ implies there exists $D^*_\mathcal{Y} \in \mathcal{B}$ such that $D^*_\mathcal{Y} \subseteq L$. So $D^*_\mathcal{Y} \subseteq L$ for each $x \in X$. But $D^*_\mathcal{Y} \subseteq \text{St}(x, \mathcal{Y})$ whence $\mathcal{Y} \prec \{ D^*_\mathcal{Y} \} \subseteq \mathcal{G}$ and hence by (ii) we have $\mathcal{G} \subseteq \mu$.

**Remark.** Thus we see that the local uniform covers describe a local uniformity as well as its entourages. We call $(X, \mu)$ a covering locally uniform space. Rest of the theory of covering locally uniform spaces regarding defining of bases, subbases, subspaces and products can easily be done parallel to the theory of covering uniform spaces. We leave that as a simple exercise.
3. Compact sets and local uniform spaces

THEOREM 3.1. In a locally uniform space \((X, \mathcal{D})\) let \(A\) be a compact set and \(B\) a closed set such that \(A \cap B = \emptyset\) then there is an entourage \(u \in \mathcal{D}\) such that \(u \cap A \times B = \emptyset\).

PROOF. Let \((X, \mathcal{D})\), \(A\) and \(B\) be as above. Since \(A \cap B = \emptyset\), \(A \subset X - B\), so for each \(a \in A\) there is a \(V_a \in \mathcal{D}\) such that \(V_a \cap X - B\). Now \(a \in A\), \(V_a \in \mathcal{D}\) there exists an \(u_a \in \mathcal{D}\) such that \(u_a \cap V_a \cap X - B\). Now \(\{u_a \cap V_a : a \in A\}\) is an open cover of \(A\) which is compact and hence it has a finite subcover, say, \(\{u_{a_i} \cap V_{a_i} : i = 1, 2, \ldots, n\}\).

Now put \(u = \bigcap_{i=1}^{n} u_{a_i}\) then \(u \in \mathcal{D}\) and it can be easily seen that \(u \cap A \times B = \emptyset\).

COROLLARY 3.2. In a locally uniform space \((X, \mathcal{D})\) let \(A\) be a closed set, \(B\) be a compact set such that \(A \cap B = \emptyset\) then there exists a \(u \in \mathcal{D}\) such that \(u[B] \cup A = \emptyset\).

THEOREM 3.3. In a locally uniform space \((X, \mathcal{D})\) if \(A\) is a compact subset of \(X\) then the family \(\{u[A] : u \in \mathcal{D}\}\) forms a fundamental system of neighbourhoods.

PROOF. Let \(N\) be any open neighbourhood of \(A\). Let \(B = X - N\). \(B\) is then closed and \(A \cap B = \emptyset\), hence, by corollary 3.2 there exists an entourage \(u \in \mathcal{D}\) such that \(u[A] \cap B = \emptyset\) implying that \(A \subset u[A] \subset X - B = N\). Hence the theorem is proved.

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