A NOTE ON PERIPHERALLY $\mathfrak{M}$-PARACOMPACT SPACES

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In [1] E.E. Grace introduced the concept of peripherally paracompact spaces. In the present paper we introduce and study peripherally $\mathfrak{M}$-paracompact spaces. Also, by making use of some other concepts introduced by E.E. Grace [1], we obtain some characterisations of $\mathfrak{M}$-paracompact spaces. A result due to D.R. Traylor [4] for paracompactness in regular spaces, has also been extended to $\mathfrak{M}$-paracompactness in normal spaces.

DEFINITION 1. A family $\mathcal{A}$ of open subsets of a space $X$ is said to have property $\mathcal{P}$ in the strong sense (resp. in the weak sense) if $\mathcal{A}$ has the property $\mathcal{P}$ as a collection of open sets in $X$ (resp. in the subspace $\cup\{A : A \in \mathcal{A}\}$ of $X$).

DEFINITION 2. A space $X$ is said to be peripherally $\mathfrak{M}$-paracompact in the strong sense (resp. in the weak sense) if for each frontier set (that is, each nowhere dense, closed set) $F$ in $X$ and each open covering $\mathcal{U}$ of $X$ of cardinality $\leq \mathfrak{M}$, there is an open refinement $\mathcal{V}$ of $\mathcal{U}$, covering $F$, which is locally finite in the strong sense (resp. in the weak sense).

THEOREM 1. A space $X$ is $\mathfrak{M}$-paracompact if and only if it is peripherally $\mathfrak{M}$-paracompact in the strong sense.

PROOF. Only the if part need be proved. Let $\mathcal{G}$ be any open covering of $X$ of cardinality $\leq \mathfrak{M}$. Let $\mathcal{H}$ be a family of mutually disjoint open sets refining $\mathcal{G}$ such that $H^* = \bigcup\{H : H \in \mathcal{H}\}$ is dense in $X$. Then, $X \sim H^*$ is a nowhere closed set. Let $\mathcal{F}$ be a locally finite, open refinement of $\mathcal{G}$ covering the frontier set $X \sim H^*$ and let $\mathcal{A}$ be a locally finite, open refinement of $\mathcal{F}$ covering the boundary of $E^* = \bigcup\{E : E \in \mathcal{F}\}$. Consider now, the family $\mathcal{H} = \{H \cap (X - E^*) : H \in \mathcal{H}\}$. It is easy to verify that $\mathcal{H}$ is a discrete family of open sets and that $\mathcal{H} \cup \mathcal{F} \cup \mathcal{A}$ is a locally finite open refinement of $\mathcal{G}$ which covers $X$ and hence $X$ is $\mathfrak{M}$-paracompact.

THEOREM 2. A normal space $X$ is peripherally $\mathfrak{M}$-paracompact in the strong sense iff it is peripherally $\mathfrak{M}$-paracompact in the weak sense.
PROOF. Let $\mathcal{C}$ be any open covering of $X$ of cardinality $\leq \mathfrak{m}$ and let $F$ be any frontier subset of $X$. If $X$ is peripherally $\mathfrak{m}$-paracompact in the weak sense, then there exists an open refinement $\mathcal{H}$ of $\mathcal{C}$ covering $F$ which is locally finite at each point of $H^* = \bigcup \{H, H \in \mathcal{H}\}$. Since $X$ is normal, and $F$ and $X \sim H^*$ are disjoint closed sets, therefore exists an open set $W : F \subseteq W \subseteq X \sim H^*$. Let $\mathcal{W} = \{W \cap H, H \in \mathcal{H}\}$. Then $\mathcal{W}$ is a locally finite open refinement of $\mathcal{C}$ which covers $F$ and hence $X$ is peripherally $\mathfrak{m}$-paracompact in the strong sense.

DEFINITION 3. A family $\mathcal{F}$ of continuous functions on a space $X$ into the non-negative real numbers is called a partition of unity on $X$ if for each point $x \in X, \sum f(x) = 1$. $\mathcal{F}$ is said to be subordinated to a covering $\mathcal{U}$ of $X$ if for each $f \in \mathcal{F}, f(X \sim U) = \{0\}$ for some $U \in \mathcal{U}$.

THEOREM 3. A normal space $X$ is $\mathfrak{m}$-paracompact iff for every open covering $\mathcal{C}$ of $X$ of cardinality $\leq \mathfrak{m}$ and for every frontier set $F$, there exists an open refinement $\mathcal{H}$ of $\mathcal{C}$, covering $F$ and which has a partition of unity subordinated to it in the weak sense.

PROOF. To prove the 'if' part, let $\mathcal{C}$ be any open covering of $X$ of cardinality $\leq \mathfrak{m}$. Let $\mathcal{H}$ be a family of disjoint open sets refining $\mathcal{C}$ such that $H^* = \bigcup \{H, H \in \mathcal{H}\}$ is dense in $X$. Then $X \sim H^*$ is a frontier set. By hypothesis, there exists an open refinement $\mathcal{W}$ of $\mathcal{C}$ which covers $X \sim H^*$ and which has a partition of unity $\Phi$ subordinated to it in the weak sense. Since $X$ is normal, and $X \sim H^*$ and $X \sim \bigcup \{W, W \in \mathcal{W}\}$ are disjoint closed sets, therefore, there exists a continuous function $g : X \rightarrow [0, 1]$ such that $g(X \sim H^*) = \{1\}$ and $g(X \sim \bigcup \{W, W \in \mathcal{W}\}) = \{0\}$. For each $f \in \Phi$, let $f(x) = f(x) \cdot g(x)$ for $x \in \bigcup \{W, W \in \mathcal{W}\}$ and let $f'(x) = 0$ for $x \in X \sim \bigcup \{W, W \in \mathcal{W}\}$. For each $H \in \mathcal{H}$, there exists a continuous function: $g_H : X \rightarrow [0, 1]$ such that $g_H(X \sim H) = \{0\}$ and $g_H(H \sim g^{-1}(0)) = \{1\}$. Let $h$ be defined as

$$h(x) = \begin{cases} \sum_{f \in \Phi} f(x), & \text{if } x \in X \sim H^* \\ \sum_{f \in \Phi} f(x) + g_H(x), & \text{if } x \in H^*. \end{cases}$$

Then $\mathcal{C}$ has the partition of unity $\Phi = \{f/h, f \in \Phi \cup \{g_H/h, H \in \mathcal{H}\}\}$ subordinated to it. Thus, every open covering of $X$ of cardinality $\leq \mathfrak{m}$ has a partition of unity subordinated to it and hence $X$ is $\mathfrak{m}$-paracompact [2, theorem 2]. Converse is obviously true, [2, theorem 2].
THEOREM 4. For a normal space $X$, the following are equivalent:

(a) $X$ is $\mathcal{M}$-paracompact.

(b) For every covering $\mathcal{U}$ of $X$ of cardinality $\leq \mathcal{M}$ and for each frontier set $F$ in $X$, there is an open refinement $\mathcal{V}$ of $\mathcal{U}$ covering $F$, such that $\mathcal{V}$ is cushioned in $\mathcal{U}$ in the strong sense.

(c) For every open covering $\mathcal{U}$ of $X$ of cardinality $\leq \mathcal{M}$ and for each frontier set $F$ in $X$, there is an open refinement $\mathcal{V}$ of $\mathcal{U}$ covering $F$, such that $\mathcal{V}$ is cushioned in $\mathcal{U}$ in the weak sense.

(d) For every open covering $\mathcal{U}$ of $X$ of cardinality $\leq \mathcal{M}$ and for each frontier set $F$ in $X$, there is an open refinement $\mathcal{V}$ of $\mathcal{U}$ covering $F$, such that $\mathcal{V}$ is $\sigma$-cushioned in $\mathcal{U}$ in the weak sense.

(e) For each every open covering $\mathcal{U}$ of $X$ of cardinality $\leq \mathcal{M}$ and for each frontier set $F$ in $X$, there is an open refinement $\mathcal{V}$ of $\mathcal{U}$ covering $F$, such that $\mathcal{V}$ is $\sigma$-cushioned in $\mathcal{U}$ in the strong sense.

PROOF. (a) $\implies$ (b). Every open covering $\mathcal{U}$ of $X$ of cardinality $\leq \mathcal{M}$ will have an open, cushioned refinement in view of Theorem 1 and hence (b) is true.

(b) $\implies$ (c) Obvious

(c) $\implies$ (d) Obvious

(d) $\implies$ (e). Since $X$ is normal, a proof similar to theorem 2 applies.

(e) $\implies$ (a). This follows in a manner similar to the proof of theorem 1.

DEFINITION 4. A space $X$ is said to be $\mathcal{M}$-paracompact in a discrete peripheral sense if for every open covering $\mathcal{U}$ of $X$ of cardinality $\leq \mathcal{M}$ there exists an open refinement $\mathcal{V}$ of $\mathcal{U}$ such that if $\mathcal{F}$ be any discrete family of closed set refining $\mathcal{V}$, then the boundary of $\bigcup \{ F : F \in \mathcal{F} \}$ is $\mathcal{M}$-paracompact with respect to the space $X$.

DEFINITION 5. A space $X$ is said to be subparacompact if for every open covering $\mathcal{G}$ of $X$, there exists a sequence $\{ \mathcal{F}_i : i = 1, \ldots \}$ of discrete families of closed sets such that $\bigcup_{i=1}^{\infty} \mathcal{F}_i$ is a refinement of $\mathcal{G}$.

THEOREM 4. If $X$ is a normal, subparacompact space which is countably paracompact in a discrete peripheral sense, then $X$ is countably paracompact.

PROOF. Essentially the same as that of ([4], theorem 5) Traylor states the theorem with 'semi-method' instead of 'subparacompact'. However, while
proving the theorem, only subparacompactness is being used. It should be noted that every normal, semi-metric space is perfectly normal and a perfectly normal space is always countably paracompact. So the theorem becomes obvious with subparacompact replaced by semi-metric.

**THEOREM 5.** If $X$ is a normal, subparacompact space which is $\mathfrak{B}$-paracompact in a discrete peripheral sense, then $X$ is $\mathfrak{B}$-paracompact.

**PROOF.** Since $X$ is $\mathfrak{B}$-paracompact in a discrete peripheral sense, therefore, $X$ is countably paracompact in a discrete peripheral sense. Then $X$ is countably paracompact by theorem 4. Now, let $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ be any open covering of $X$ of cardinality $\leq \aleph_\omega$. Let $A$ be well ordered by $\prec$. Let $\mathcal{U}'$ be an open refinement of $\mathcal{U}$ covering $X$ such that the boundary of the union of each discrete family of closed sets refining $\mathcal{U}'$ is $\mathfrak{B}$-paracompact with respect to $X$. Since $X$ is subparacompact, there exists a sequence $\{\mathcal{F}_i : i \in \mathbb{N}\}$ of discrete families of closed sets. For each $\alpha \in A$, let $\mathcal{F}_1\alpha$ denote the subfamily of $\mathcal{F}_1$ consisting of all sets $G \in \mathcal{F}_1\alpha$ for which $\alpha$ is the first index such that $G \subseteq U_\alpha$. If $G \in \mathcal{F}_1\alpha$ for some $\alpha$, denote by $V_\alpha$ an open set which contains boundary of $G$ such that $V_\alpha \supset U_\alpha$ and $V_\alpha$ does not intersect $[(\cup \{F : F \in \mathcal{F}_1\}) \sim G]$. Denote by $\mathcal{F}_1\alpha$ the family consisting of all sets $V$ such that there exists $G \in \mathcal{F}_1\alpha$ such that $V = V_\alpha$. Since boundary of $\cup \{F : F \in \mathcal{F}_1\}$ is $\mathfrak{B}$-paracompact and $\mathcal{F}_1\alpha = \cup \mathcal{F}_1\alpha$ is a covering of the boundary of $\cup \{F : F \in \mathcal{F}_1\}$; therefore, there exists a locally finite open refinement $\mathcal{F}_1'$ of $\mathcal{F}_1$ such that $\mathcal{F}_1'$ covers boundary of $\cup \{F : F \in \mathcal{F}_1\}$. Now, denote by $\mathcal{F}_2$ the family consisting of all sets $V$ for which there is a $G \in \mathcal{F}_1\alpha$ such that $x \in V$ iff either $x \in G$ or $x$ is a point of a member of $\mathcal{F}_1\alpha$ which intersects $G$. Clearly, $\mathcal{F}_2$ is an open refinement of $\mathcal{U}'$ which covers $\cup \{F : F \in \mathcal{F}_1\}$. Now consider $\mathcal{F}_2\alpha$. Denote by $\mathcal{F}_2\alpha$ the family consisting of all sets $G$ such that there exists $H \in \mathcal{F}_2\alpha$ such that $G = H \sim [H \cap (\cup \{V : V \in \mathcal{F}_1\})]$. Clearly, $\mathcal{F}_2\alpha$ is discrete family of closed sets refining $\mathcal{U}'$. For each $\alpha \in A$, denote by $\mathcal{F}_2\alpha$ the subfamily of $\mathcal{F}_2\alpha$ consisting of only those sets each of which is a subset of $U_\alpha$ but none is a subset of $U_\beta$ for $\beta < \alpha$. If $G \in \mathcal{F}_2\alpha$, denote by $V_\alpha$ an open set containing the boundary of $G$ such that $H_\alpha \subseteq V_\alpha$, $V_\alpha$ does not intersect $[(\cup \{F : F \in \mathcal{F}_2\}) \sim G]$. Let $\mathcal{F}_2\alpha$ denote the family consisting of all sets $V$ for which there is a $G \in \mathcal{F}_2\alpha$ such that $V = V_\alpha$. Let $\mathcal{F}_2 = \cup \mathcal{F}_2\alpha$. As before, there exists a locally finite, open refinement $\mathcal{F}_2$.
of \( Y'_2 \) which covers the boundary of \( \bigcup \{ F : F \in \mathcal{F}_2 \} \) and thus there is a locally finite open refinement \( Y''_2 \) of \( Z' \) such that \( Y''_2 \) covers \( \bigcup \{ F : F \in \mathcal{F}_2 \} \). This process may be continued indefinitely as follows: for each positive integer \( n > 2 \), denote by \( \mathcal{F}'_n \) the collection which consists of all sets \( G \) for which there is a \( H \in \mathcal{F}_n \) such that \( G = H \sim (H \cap \bigcup \{ V : V \in Y''_i, i = 1, \ldots, n-1 \}) \).

Clearly, \( \mathcal{F}'_n \) is a discrete family of closed sets such that \( \mathcal{F}'_n \) refines \( Z' \). As before, denote by \( \mathcal{F}_{2a} \) the subfamily of \( \mathcal{F}'_n \) consisting of just those sets each of which is a subset of \( U_{\alpha} \) but none is a subset of \( U_{\beta} \) for \( \beta < \alpha \). For \( G \in \mathcal{F}_{2a} ' \), let \( V_G \) denote an open set containing the boundary of \( G \) such that \( V_G \supseteq U_{\alpha} \). \( V_\alpha \) does not intersect \( \bigcup \{ F : F \in \mathcal{F}_i, i = 1, \ldots, n-1 \} \) and also does not intersect \( \bigcup \{ F : F \in \mathcal{F}_i \} \). \( Y_{mn} \) denotes the family consisting of those sets \( G \in \mathcal{F}_{ma} \) such that \( V = V_G \) and if \( \mathcal{Y}_n = \bigcup A_{ma} \), then there exists a locally finite, open refinement \( \mathcal{Y}'_n \) of \( \mathcal{Y}_n \) such that \( \mathcal{Y}'_n \) covers the boundary of \( \bigcup \{ F : F \in \mathcal{F}_n \} \) and thus there is a locally finite open refinement \( \mathcal{Y}''_n \) of \( Z' \) such that \( \mathcal{Y}''_n \) covers \( \bigcup \{ F : F \in \mathcal{F}_n \} \). Now, \( \bigcup_{n=1}^{\infty} \mathcal{Y}''_n \) is a \( \sigma \)-locally finite, open refinement of \( Z' \) and hence of \( Z' \). Thus every open covering of \( X \) of cardinality \( \leq \mathfrak{m} \) has a \( \sigma \)-locally finite open refinement. Also, \( X \) is a countably paracompact. Therefore \( X \) is \( \mathfrak{m} \)-paracompact (\[3\], theorem 5).

**Corollary** Every normal space which is either semi-metric or developable or Moore, and is \( \mathfrak{m} \)-paracompact in a discrete peripheral sense, is \( \mathfrak{m} \)-paracompact.

**Proof.** Every semi-metric, or developable, or Moore space is subparacompact and hence the result follows from theorem 5.

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REFERENCES


