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ON REFLECTIVE SUBCATEGORIES

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Let $\mathscr{A}$ be a given full subcategory of $\mathscr{C}$. Under what conditions does there exist a smallest, replete, reflective subcategory $\mathscr{A}^*$ of $\mathscr{C}$ for which $\mathscr{A}^*$ contains $\mathscr{A}$? The answer to this question is unknown. Even if $\mathscr{C}$ is the category of uniform spaces and uniform maps, it is an open question. J. F. Kennison answered to the above question partially, in case $\mathscr{C}$ is a complete category with a well-founded bicategory structure and $\mathscr{A}$ is a full, replete and closed under the formation of products [1]. In this paper, the author could construct a smallest, replete, reflective subcategory $\mathscr{A}^*$ of $\mathscr{C}$ for which $\mathscr{A}^*$ contains $\mathscr{A}$ under other conditions except the fact that $\mathscr{A}$ is full.

Throughout this paper, our definitions and notations are based on ([1], [5], [6], [10], [11]). The class of all epimorphisms of $\mathscr{C}$ will be denoted by $E_{\mathscr{C}}$ and $M_{\mathscr{C}}$ the class of all monomorphisms of $\mathscr{C}$.

DEFINITION. Let $\mathscr{C}$ be a category and let I and P be classes of morphisms on $\mathscr{C}$. Then (I,P) is a *bicategory structure* on $\mathscr{C}$ provided that:

B-1) Every isomorphism is in $I\cap P$.

B-2) I and P are closed under the composition of morphisms.

B-3) Every morphism f can be factored as $f=f_1 f_0$ with $f_1\in I$ and $f_0\in P$. Moreover this factorization is unique to within an isomorphism in the sense that if $f=gh$ and $g\in I$ and $h\in P$ then there exists an isomorphism e for which $ef_0=h$ and $ge=f_1$.

B-4) $P\subset E$

B-5) $I\subset M$

DEFINITION. (I,P) is a *right bicategory structure* on $\mathscr{C}$ if it satisfies B-1, B-2, B-3 and B-4.

DEFINITION. Let $\mathscr{C}$ be a category and let I and P be classes of morphisms of $\mathscr{C}$. Let $X\in\mathscr{C}$. Then A is an I-subobject of X if $I\cap\mathrm{Hom}(A,X)\neq\phi$. The category $\mathscr{C}$ is *I-well-powered* if each $X\in\mathscr{C}$ has a representative set of I-subobjects. The terms *P-quotient* and *P-co-well-powered* are defined dually.

DEFINITION. A bicategory sutrcture (I, P) on $\mathscr{C}$ is *well-founded* if $\mathscr{C}$ is I-well-powered and P-co-well-powered.

DEFINITION. A subcaterory $\mathfrak{A}$ of $\mathscr{C}$ is *replete* if $A \in \mathfrak{A}$ and X isomorphic to A imply $X \in \mathfrak{A}$.

DEFINITION. Let $\mathfrak{A}$ be a subcategory of $\mathscr{C}$. A morphism $f: X \longrightarrow Y$ is *injective* with respect to $\mathfrak{A}$ if for all morphisms $g: X \longrightarrow A$ with $A \in \mathfrak{A}$ there exists a morphism $h: Y \longrightarrow A$ such that $hf = g$. The class of all morphisms of $\mathscr{C}$ which are injective with respect to $\mathfrak{A}$ is denoted by $\Psi_{\mathscr{C}}(\mathfrak{A})$.

LEMMA 1. *Let $\mathscr{C}$ be a complete category and let $\mathfrak{A}$ be a full, reflective subcategory of $\mathscr{C}$. Then $\mathfrak{A}$ is a complete category.*

DEFINITION 1. Let $\mathfrak{A}$ be a subcategory of a complete category $\mathscr{C}$ and let $L\mathfrak{A}$ be the smallest limit closed subcategory of $\mathscr{C}$ such that $\mathfrak{A} \subset L\mathfrak{A}$. Then $L\mathfrak{A}$ is called the *limit closure of $\mathfrak{A}$*.

LEMMA 2. (Freyd-Isbell) *Let $\mathscr{C}$ be a category with products. Let (I, P) be a right bicategory structure on $\mathscr{C}$ such that $\mathscr{C}$ is P-co-well-powered. Then $\mathfrak{A}$ is P-reflective iff $\mathfrak{A}$ is closed under the formation of products and I-subobjects.*

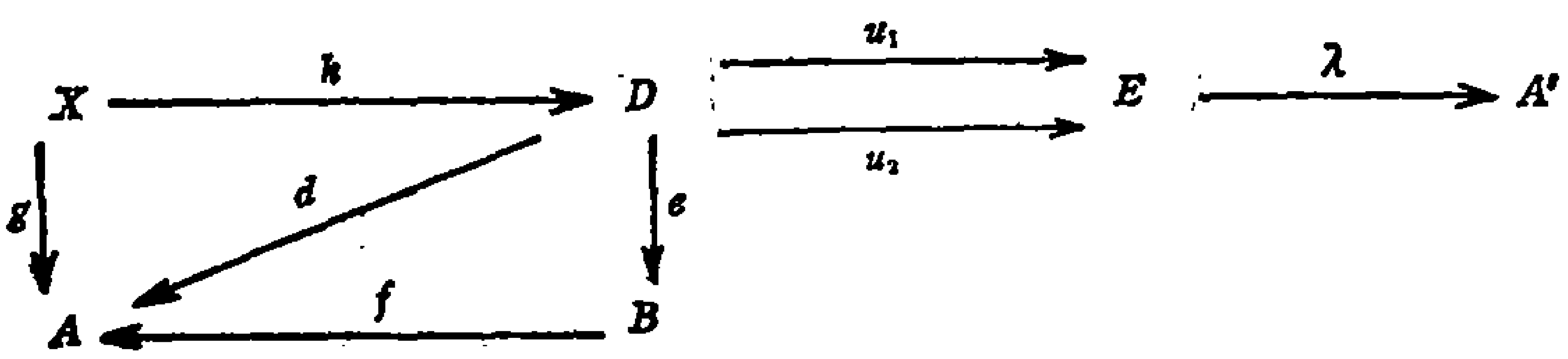
PROOF. For a proof, see (1.2) in [1].

THEOREM 3. *Let $\mathscr{C}$ be complete and let (I_0, P_0) be a well-founded bicategory structure on $\mathscr{C}$, $\mathfrak{A}$ a full subcategory of $\mathscr{C}$, $\mathscr{B}$ the full subcategory consisting of I_0-subobjects of the limit closure $L\mathfrak{A}$ of $\mathfrak{A}$, and $P = \Psi_{\mathscr{B}}(L\mathfrak{A}) \cap F_{\mathscr{B}}$. Assume that $\mathscr{B}$ is P-co-well-powered. Let I be a class of morphisms such that (I, P) is a right bicategory structure on $\mathscr{B}$ and let $\mathfrak{A}^*$ be the full subcategory of all I-subobjects of members of $L\mathfrak{A}$. Then $\mathfrak{A}^*$ is the smallest, full, replete, reflective subcategory containing $\mathfrak{A}$.*

REMARK. Let (I_0, P_0) be a well-founded bicategory structure on $\mathscr{C}$ and let $I = \{g : g = fe$ and $e \in P$ implies e is an isomorphism$\}$. Then I is unique and (I, P) is a right bicategory structure on $\mathscr{B}$ under the hypothesis of Theorem 2.3[1].

PROOF OF THEOREM 3. According to Lemma 2.1 any full, reflective subcategory of the complete category $\mathscr{C}$ is complete. Thus any full, reflective subcategory of $\mathscr{C}$ containing $\mathfrak{A}$ contains the limit closure $L\mathfrak{A}$. It suffices to show that

$\mathcal{A}^*$ is the smallest, full, replete, reflective subcategory of $\mathcal{C}$ for which $L\mathcal{A}\subset\mathcal{A}^*$. Also $L\mathcal{A}$ is replete and closed under the formation of products since $L\mathcal{A}$ is closed under limits. $\mathcal{A}^*$ contains $L\mathcal{A}$ by its construction. By Lemma 2.2, $\mathcal{A}^*$ is reflective in $\mathcal{B}$ and $\mathcal{B}$ is also reflective in $\mathcal{C}$. Thus $\mathcal{A}^*$ is reflective in $\mathcal{C}$. Let $\mathcal{X}$ be the full, replete, reflective subcatogory of $\mathcal{C}$ containing $\mathcal{A}$. Then $\mathcal{X}$ contains $L\mathcal{A}$. To prove $\mathcal{A}^*\subset\mathcal{X}$, let $x\in\mathcal{A}^*$ be given. Then there exists $g: X\longrightarrow A$ with $g\in I$ and $A\in L\mathcal{A}$. Let $h: X\longrightarrow D$ be a reflection map in $\mathcal{X}$. It suffices to show that h is an isomorphism. Since $A\in L\mathcal{A}\subset\mathcal{X}$ there exists $d: D\longrightarrow A$ such that $g=dh$. Let $e: D\longrightarrow B$ be a reflection map in $\mathcal{B}$ then there exists unique morphism $f: P\longrightarrow A$ such that $fe=d$. Then $e\in P_0$. Indeed, let $\{f_\alpha : D\longrightarrow B_\alpha\}$ be a representative set of morphisms such that $B_\alpha\in\mathcal{B}$ and $f_\alpha\in P_0$. Let $f: D\longrightarrow \Pi B_\alpha$ be determined by $P_\alpha f=f_\alpha$ for all α. Let $f=f_1 f_0$ in (I_0, P_0). Then f_0 is the required morphism e. Again since $feh=g\in I$, $eh\in I$.



In order to show $h\in E_{\mathcal{X}\cup\mathcal{B}}$, assume $u_1 h=u_2 h$ where u_1, $u_2: D\longrightarrow E$ and $E\in\mathcal{X}\cup\mathcal{B}$. If $E\in\mathcal{X}$, then $u_1=u_2$ since h is a reflection map. If $Y\in\mathcal{B}$ there exists $\lambda: E\longrightarrow A'$ with $\lambda\in I_0$ and $A'\in L\mathcal{A}$. Thus $\lambda u_1 h=\lambda u_2 h$. Since $A'\in\mathcal{X}$ and h is a reflection map and λ is a monomorphism, $u_1=u_2$. Since e, h are reflection maps and $L\mathcal{A}\subset\mathcal{X}$ and $L\mathcal{A}\subset\mathcal{B}$, $eh\in\Psi_{\mathcal{B}}(L\mathcal{A})$. Since e is an epimorphism and h is an epimorphism in $\mathcal{X}\cup\mathcal{B}$, eh is an epimorphism in $\mathcal{B}$. Thus $eh\in P$, and $eh\in I\cap P$. Hence eh is an isomorphism and $(eh)^{-1}(eh)=id$, $h(eh)^{-1}eh=h$, $h\in E_{\mathcal{X}\cup\mathcal{B}}$. It follows that h is an isomorphism.

COROLLARY 4. *Let $\mathcal{C}$ be a complete category with well-founded bicategory structure (I, P) and let $\mathcal{A}$ be any full subcategory of $\mathcal{C}$. Let $\mathcal{A}'$ be the smallest, full, replete subcategory which is closed under products containing $\mathcal{A}$. Let $\mathcal{B}'$ be the full subcategory consisting of all I-subobjects of $\mathcal{A}'$. Let $\mathcal{B}$ be the full subcategory consisting of all I-subobjects of $L\mathcal{A}$. Then $\mathcal{B}=\mathcal{B}'$.*

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