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CHARACTERIZATIONS OF R-COMPACT ORDERED SPACES

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In [1], the following problem was raised. What are characterizations of R-compact ordered spaces and R-compact topological lattices? In this paper, we give a few external characterizations of them, which are analogous to real compact topological spaces [3]. For the terminology, we refer to [1, 5].

THEOREM 1. *Let* $(X, \leq, O)$ *be a completely regular ordered space. Then* X *is an* R*-compact ordered space if and only if there does not exist a completely regular ordered space* $(X', \leq', O')$ *satisfying the following two conditions.*

(i) *There exists an iseomorphism* e *from* X *into* X' *such that* $e(X) \neq \overline{e(X)} = X'$, *where* $\overline{e(X)}$ *denotes the closure of* $e(X)$ *in* X'.

(ii) *For every continuous isotone* $f: X \to R$, *there is continuous isotone* $\bar{f}: X' \to R$ *such that* $\bar{f} \circ e = f$, *where* R *denotes the real line equipped with the usual order and topology.*

LEMMA 2. *Let* A *be a subspace of a topological partially ordered space* $(X, \leq, O)$. *If every continuous isotone* $g: A \to R$, *such that either* $g(a) \geq 1$ *for all* $a \in A$ *or* $g(a) \leq -1$ *for all* $a \in A$, *has a continuous isotone extension on* X *(that is, there exists a continuous isotone* $\bar{g}: X \to R$ *with* $\bar{g}|A = g$), *then every continuous isotone* $f: A \to R$ *has a continuous isotone extension on* X.

LEMMA 3. *Let* A *be a subspace of a topological partially ordered space* $(X, \leq, O)$. *If every continuous isotone* $f: A \to R$ *has a continuous isotone extension on* X, *then every continuous isotone* $f: A \to R^{|S|}$ *has a continuous isotone extension on* X, *where* $|S|$ *denotes the cardinal number of a set* S.

Moreover, if $\overline{A} = X$, *then every continuous isotone* $f: A \to B = \overline{B} \subset R^{|S|}$ *has a continuous isotone extension on* X.

In [2] and [4], they constructed an ordered compactification $\beta_1 X$ for a completely regular ordered space X as follows. Let $C_1 X$ be the set of continuous isotones of X into I, where I denotes the unit interval equipped with the usual order and topology.

Define $\beta_1 : X \to I^{|C_1 X|}$ by $\beta_1(x)(f) = f(x)$ for each $f \in C_1 X$ and each $x \in X$.

Then β_1 is an iseomorphism from X into $I^{|C,X|}$. Let $\beta_1 X=\overline{\beta_1(X)}$; then $\beta_1 X$ is iseomorphic with Nachbin ordered compactification.

Combining the two lemmas, we have

THEOREM 4. *Let $(X,\leq,O)$ be a completely regular ordered space. Then X is R-compact ordered if and only if for every point $x_0\in\beta_1X-\beta_1(X)$ there exists a continuous anti-isotone $f:\beta_1X\to[-1,1]$ such that either $f(x_0)=0$ and $f(x)>0$ for all $x\in\beta_1(X)$ or $f(x_0)=0$ and $f(x)<0$ for all $x\in\beta_1(X)$, where $[-1,1]$ denotes the interval equipped with the usual order and topology.*

REMARK 5. We note that Theorem 4 holds for β_0X [1]. By using the same lines as those in Theorem 1, we can obtain an external characterization of R-compact topological lattices as follows. A completely regular topological lattice L is an R-compact topological lattice if and only if there does not exist a completely regular topological lattice L' which satisfies the following two conditions: (i) there exists an iseomorphism $e:L\to e(L)\subset L'$ such that $e(L)\neq\overline{e(L)}=L'$; (ii) for every continuous homomorphism $f:L\to R$, there is a continuous homomorphism $\bar{f}:L'\to R$ such that $\bar{f}\circ e=f$.

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