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# $c$-CONTINUOUS FUNCTIONS AND COCOMPACT TOPOLOGIES

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## 1. Introduction

In [2] there is introduced the notion of a $c$-continuous function. The function $f: X \longrightarrow Y$ is $c$-continuous if whenever $U \subset Y$ is an open set with compact complement, $f^{-1}(U)$ is open. As noted in [1], $f: X \longrightarrow Y$ is $c$-continuous if and only if the same function is continuous after we retopologise $Y$ with the topology having as a basis

$$\{U \subset Y \mid U \text{ is open and } Y-U \text{ is compact}\}.$$

In this note, we study the operator which changes the topology in this way, and show that many of the properties described in [2] and [3] fit into a broader context.

It would seem that the greatest use for $c$-continuous functions may be theorems of the sort "$c$-continuous$\Rightarrow$continuous" since we have the obvious characterisation that $f: X \longrightarrow Y$ is $c$-continuous iff $f^{-1}(C)$ is closed whenever $C$ is a closed compact subset of $Y$. Thus in the presence of a "$c$-continuous$\Rightarrow$continuous" theorem, we need only apply the "inverse image of a closed set is closed" criterion to compacta to deduce continuity of a function.

Below we show that theorems of the sort "$c$-continuous$\Rightarrow$continuous" in which there are no restrictions on the domain are usually corollaries of theorems involving a retopologising of the range.

## 2. The cocompactness operator

Let $\tau$ be a topology on a set $X$. Define $c(\tau)$, the *cocompact topology,* on $X$ by

$$c(\tau) = \{\phi\} \cup \{O \in \tau \mid X-O \text{ is compact in } \tau\}.$$

It is easily verified that $c(\tau)$ is a topology on $X$.

The basic relation between the cocompactness operator and $c$-continuous functions is the following, which is theorem 1 of [1].

THEOREM 1. *$f: X \longrightarrow (Y, \tau)$ is $c$-continuous if and only if $f: X \longrightarrow (Y, c(\tau))$ is continuous.*

In this theorem, the topology on $X$ remains unchanged so is unspecified.

THEOREM 2. *Cocompact topologies are compact.*

PROOF. Let $\tau$ be a topology on $X$ and let $\sigma \subset c(\tau)$ be a cover of $X$. Choose $O \in \sigma$ with $O \neq \phi$. Then $X-O$ is compact in $\tau$, so finitely many members of $\sigma$ cover $X-O$. Hence finitely many members of $\sigma$ cover $X$; thus $c(\tau)$ is compact.

COROLLARY 3. *Let $\tau$ be a topology. Then $c(\tau)=\tau$ if and only if $\tau$ is a compact topology.*

PROOF. One implication follows from theorem 2 and the other is trivial.

The basic properties of $c$-continuous functions described in §2 of [2] are immediate consequences of theorem 1 and the corresponding properties of continuous functions. The existence of example 3 of [2] should not be surprising.

A number of theorems involving $c$-continuity, particularly in [3], involve an interplay between the cocompactness operator and a space-constructing process. This process might be restriction (considered in §3), the taking of a product (considered in §4) or the taking of a quotient (considered in §5). We find that there is a metatheorem which tells us that if $P$ is the process and $cP(\tau) \subset Pc(\tau)$, then $c$-continuity of a function $f$ whose range is topologised by $\tau$ guarantees $c$-continuity of the function $P(f)$. Thus, for example, theorem 6 tells us that if $A$ is a subset of the Hausdorff space $(X, \tau)$ then $c(\tau|A) \subset c(\tau)|A$: in this case the process $P$ is restriction of a topology to $A$. We are able to deduce that if $f: Y \longrightarrow X$ is a $c$-continuous function where $X$ is Hausdorff and $A \subset X$ is such that $f(Y) \subset A$, then the function $f: Y \longrightarrow A$ is $c$-continuous, which is a generalisation of theorem 2.13 of [3].

## 3. The behaviour of the cocompactness operator under restrictions

Example 2.12 of [3] shows that in general we cannot restrict the range of a $c$-continuous function and still have a $c$-continuous function. In terms of the cocompactness operator, this example tells us that in general it is false that $c(\tau|A) \subset c(\tau)|A$ when $A$ is a subset of $X$. The reverse inclusion is also false in general, even if $\tau$ is a Hausdorff topology.

EXAMPLE 4. Let $X=[0.1]$, $A=(0,1)$ with $\tau$ the usual topology on $X$. Then $(1/2, 1) \in \tau = c(\tau)$ by corollary 3, so that $(1/2, 1) \in c(\tau)|A$. However $A-(1/2, 1) = (0, 1/2]$ is not compact in $\tau|A$, so $(1/2, 1) \notin c(\tau|A)$.

Thus, in general, there is no connection between the compact topology $c(\tau|A)$ and the topology $c(\tau)|A$. However, under certain conditions there is a connection.

THEOREM 5. *Let $A$ be a closed subset of the topological space $(X,\tau)$. Then $c(\tau|A)=c(\tau)|A$.*

PROOF. Suppose $O\in c(\tau|A)$. Then $O\in\tau|A$ and $A-O$ is compact in $\tau|A$, hence in $\tau$. Since $O\in\tau|A, \exists P\in\tau$ with $P\cap A=O$. Since $A$ is closed under $\tau$, $X-A\in\tau$ so $P\cup(X-A)\in\tau$. But $X-[P\cup(X-A)]=A-P=A-O$, which is compact in $\tau$, so that $P\cup(X-A)\in c(\tau)$. Since $[P\cup(X-A)]\cap A=O$, we have that $O\in c(\tau)|A$, so $c(\tau|A)\subset c(\tau)|A$.

Conversely suppose $O\in c(\tau)|A$. Then $\exists P\in c(\tau)$ with $O=P\cap A$. Since $P\in c(\tau), X-P$ is compact under $\tau$, so that $A-O=A-P$, being a closed subset of the compact set $X-P$, is compact in $\tau$, hence in $\tau|A$. Further $O\in\tau|A$, so $O\in c(\tau|A)$, and so $c(\tau)|A\subset c(\tau|A)$.

Theorem 2.14 of [3] is an immediate consequence of theorem 5.

THEOREM 6. *Let $A$ be any subset of $X$ and let $\tau$ be a Hausdorff topology on $X$. Then $c(\tau|A)\subset c(\tau)|A$.*

PROOF. Suppose $O\in c(\tau|A)$. Then $O\in\tau|A$ and $A-O$ is compact in $\tau|A$, hence in $\tau$, so is closed in $\tau$. Thus $X-(A-O)\in\tau$, so $X-(A-O)\in c(\tau)$. But $[X-(A-O)]\cap A=O$. Thus $O\in c(\tau)|A$.

Theorem 2.13 of [3] is an immediate consequence of theorem 6.

Theorem 5 of [2] is a consequence of theorem 6, theorem 1 and corollary 3.

## 4. Cocompactness and products

Let $\{\tau_\alpha\}$ be a family of topologies and $\Pi\tau_\alpha$ denote their product. Then $\Pi c(\tau_\alpha)\subset c(\Pi\tau_\alpha)$, for let $\Pi O_\alpha$ be a subbasic open set of $\Pi c(\tau_\alpha)$, i.e. $O_\alpha=X_\alpha$, where $X_\alpha$ is the underlying set, unless $\alpha=\beta$, and $O_\beta\in\tau_\beta$ has compact complement. Then $\Pi O_\alpha\in\Pi\tau_\alpha$ and $\Pi X_\alpha-\Pi O_\alpha=\Pi(X_\alpha-O_\alpha)$ is compact, being the product of compacta. Hence $\Pi O_\alpha\in c(\Pi\tau_\alpha)$, and so $\Pi c(\tau_\alpha)\subset c(\Pi\tau_\alpha)$.

In general the reverse inclusion is false, cf example 2.2 of [3]. However we do have the following extension of theorem 2.1 of [3].

THEOREM 7. *Let $\{\tau_\alpha\}$ be a collection of locally compact Hausdorff topologies. Then $c(\Pi\tau_\alpha)=\Pi c(\tau_\alpha)$.*

PROOF. By the above comments we need to show that $c(\Pi\tau_\alpha)\subset\Pi c(\tau_\alpha)$. Suppose $x\in O\in c(\Pi\tau_\alpha)$. It suffices to find $U_\alpha\in\tau_\alpha$ so that $U_\alpha=X_\alpha$ for all but finitely many indices $\alpha$, $X_\alpha-U_\alpha$ is compact for all $\alpha$ and $x\in\Pi U_\alpha\subset O$.

Now $\exists\alpha_1\cdots,\alpha_n$ and $O_{\alpha_i}\in\tau_{\alpha_i}$ so that

$$x\in\bigcap_{i=1}^{n}\pi_{\alpha_i}^{-1}(O_{\alpha_i})\subset O,$$

where $\pi_\beta:\Pi X_\alpha\longrightarrow X_\beta$ denotes the projection.

For each $y\in X-O$, choose sets $K_y$ and $L_y$ as follows: for some $i$, $y\notin\pi_{\alpha_i}^{-1}(O_{\alpha_i})$, so $y_{\alpha_i}\neq x_{\alpha_i}$. Since $X_{\alpha_i}$ is locally compact and Hausdorff, $y_{\alpha_i}$ has a closed compact neighbourhood $K_y$ in $X_{\alpha_i}$ for which $x_{\alpha_i}\notin K_y$. Let $L_y=\pi_{\alpha_i}^{-1}(K_y)$. Then $L_y$ is a neighbourhood of $y$ and $x\notin L_y$.

Now $\{\mathrm{Int}L_y|y\in X-O\}$ is an open cover of the compact set $X-O$. Let $\{L_{y_i}|i=1,\cdots,l\}$ be a cover of $X-O$.

For each $\alpha$, let $K_\alpha=\bigcup\{K_{y_i}|K_{y_i}\subset X_\alpha\}$. This is a finite union of closed compacta, so for each $\alpha$, $K_\alpha$ is a closed compact set. Furthermore, $K_\alpha=\phi$ for all but finitely many indices $\alpha$. Let $U_\alpha=X_\alpha-K_\alpha$. Then $U_\alpha\in c(\tau_\alpha)$ and $U_\alpha\neq X_\alpha$ for only finitely many indices $\alpha$. Further, $x_\alpha\in U_\alpha$. Thus $x\in\Pi U_\alpha\in O$.

## 5. Cocompactness and quotients

Let $\tau$ be a topology and $\sim$ an equivalence relation on a set $X$. We consider in this section the relationship between $c(\tau/\sim)$ and $c(\tau)/\sim$.

THEOREM 8. *Let $\tau$ be a topology and $\sim$ an equivalence relation on the same set. Then $c(\tau)/\sim\subset c(\tau/\sim)$.*

PROOF. Suppose the set is $X$. Let $\pi:X\longrightarrow X/\sim$ denote the canonical projection, i.e. $\pi(x)$ is the equivalence class of $x$ under $\sim$. Recall that the topology $\tau/\sim$ is defined on $X/\sim$ by $A\subset X/\sim$ is $(\tau/\sim)$-open if and only if $\pi^{-1}(A)\in\tau$. We have the following:

$A\in c(\tau)/\sim$

$\Longleftrightarrow \pi^{-1}(A)\in c(\tau)$

$\Longleftrightarrow \pi^{-1}(A)\in\tau$ and $X-\pi^{-1}(A)$ is $\tau$-compact.

$\Longleftrightarrow A\in\tau/\sim$ and $(X/\sim)-A$ is $(\tau/\sim)$-compact.

Compactness of $(X/\sim)-A$ follows from compactness of $X-\pi^{-1}(A)$ and the equation $(X/\sim)-A=\pi(X-\pi^{-1}(A))$. Thus $A\in c(\tau/\sim)$, so $c(\tau)/\sim\subset c(\tau/\sim)$.

The reverse inclusion, which is what is needed for the metatheorem discussed in 2, is false in general. To obtain the reverse inclusion, we need to be able to reverse the last implication in the proof of theorem 8, i.e. show that if $(X/\sim)-A$ is compact then so is $\pi^{-1}((X/\sim)-A)=X-\pi^{-1}(A)$. If $\pi$ were a proper map then we would be able to deduce this. However, even if $\pi$ were at most two-to-one, we still cannot deduce that $\pi$ is proper as the following example shows.

EXAMPLE 9. Let $X=(-1, 1)$, $\tau$ the usual topology and let $\sim$ be the equivalence relation generated by $t\sim t+1$ for $t\in(-1,0)$. Then $(X/\sim, \tau/\sim)$ is just the circle, so is compact. However $X$ itself is not compact, so the canonical projection is not proper.

This example also shows that, in general, $c(\tau)/\sim$ and $c(\tau/\sim)$ are not equal. As already noted, $\tau/\sim$, and hence $c(\tau/\sim)$, is just the usual topology on the circle. However, every non-empty $c(\tau)$-open set must contain $\{t\in(-1, 1)\,|\,|t|>r\}$ for some $r\in(0,1)$. Thus every non-empty $(c(\tau)/\sim)$-open set must contain a deleted neighbourhood of $O\in X/\sim$ in the usual topology. Thus in this case $c(\tau/\sim)$ $\not\subset c(\tau)/\sim$.

## 6. Connections with sequences

Let $\tau$ be a topology on a set $X$, let $(x_n)$ be a sequence in $X$ and $x\in X$. Say that $(x_n)$ *converges to* $x$ *in* $\tau$ and write $x_n\xrightarrow{\tau}x$ if $\forall\tau$-neighbourhood $N$ of $x$, $\exists$ a natural number $n_0$ so that whenever $n\geq n_0$, $x_n\in N$.

THEOREM 10. *Let $\tau$ be a topology on $X$ in which compacta are closed and let $(x_n)$ be a sequence in $X$ converging to $x$ in $\tau$. If $y\in X$ and $x_n\xrightarrow{c(\tau)}y$ then $x=y$.*

PROOF. Suppose $y\in X$ and $x\neq y$. Now $\{y\}$ is compact, therefore closed in $\tau$, so $X-\{y\}$ is a $\tau$-neighbourhood of $x$. Hence $\exists n_0$ so that $\forall n\geq n_0$, $x_n\in X-\{y\}$. Let

$$K=\{x\}\cup\{x_n\,|\,n\geq n_0\}.$$

Then $K$ is compact, hence closed, in $\tau$. Thus $X-K\in c(\tau)$. But then $X-K$ is a $c(\tau)$-neighbourhood of $y$ containing none of the tail of the sequence $(x_n)$. Thus $(x_n)$ cannot converge to $y$ in $c(\tau)$.

COROLLARY 11. *If all compacta in a space are closed. then no sequence converges to more than one point.*

PROOF. If $x_n\xrightarrow{\tau}x$ then $x_n\xrightarrow{c(\tau)}x$, so the result follows from theorem 10.

Theorem 2.9. of [3] is another corollary of theorem 10: in fact the space $Y$ need not be Hausdorff; instead we require only that compacta be closed. If $\tau$ denotes the topology on $Y$ and $(x_n)$ is a sequence in $X$ converging to $p\in X$, then $c$-continuity of $f$ tells us that $f(x_n)\xrightarrow{c(\tau)}f(p)$. If also $f(x_n)\xrightarrow{\tau}y$ then by theorem 10, $y=f(p)$.

The following example shows that the closedness of compacta under $\tau$ does not in general imply uniqueness of limits of sequences in $c(\tau)$.

EXAMPLE 12. Let $X=\boldsymbol{R}$, $\tau$=usual topology. Then $\tau$ is Hausdorff so compacta are closed. The sequence $(n)$ coverges to $x$ in $c(\tau)$ $\forall x\in\boldsymbol{R}$, for if $x\in\boldsymbol{R}$ and $x\in O\in c(\tau)$, then in $\tau$, $\boldsymbol{R}-O$ is compact, hence bounded. Thus $\exists n_0$ so that $\forall n\geq n_0$, $n\in O$. Thus $n\xrightarrow{c(\tau)}x$.

## 7. Conclusions

The above results suggest that many theorems involving $c$-continuous functions in which there are no restrictions on the domain are really theorems involving a change in the topology on the range space. Loosely speaking, we might say that the wrong topology has been imposed on the range. However, as indicated in the introduction, it might be desirable in a particular instance to impose this "wrong" topology.

Similar remarks can also be made about several other versions of non-continuous functions, for example, $c^*$-continuous functions studied in [3] and [4] (cf theorem 3.1 of [3]), and almost-continuous functions studied in [5].

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